Optimal control problems and some applications to inverse problems

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Outline

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Optimal control problems

Consider the optimal control problem to minimize the cost functional

$$J(u, z) = \frac{1}{2}\|u - \overline{u}\|^2_{\mathcal{V}} + \frac{\alpha}{2}\|z\|^2_{\mathcal{U}}$$

subject to

$$u = \mathcal{H}z + u_p$$

and to pointwise control constraints

$$z \in \mathcal{U}_{ad} = \{w \in \mathcal{U} \colon z_1 \leq w \leq z_2\}.$$

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$$z \in \mathcal{U}_{ad} = \{w \in \mathcal{U} \colon z_1 \leq w \leq z_2\}.$$

Here $\overline{u}, z_1, z_2$ are given, $\alpha \in \mathbb{R}_+$, $\mathcal{U}$ is a control space, and

$$\|z\|_{\mathcal{U}}^2 := \langle Az, z\rangle_{L_2}, \quad A : \mathcal{U} \to \mathcal{U}'.$$

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- Numerical experiments
- Inverse and ill-posed problems: Conjugate gradient methods

Boundary control problems

Parabolic boundary control problem

Let

$$Q = \Omega \times (0, T), \quad \Sigma = \Gamma \times (0, T), \quad T > 0.$$

The solution operator: $u(x, T) = \mathcal{H}z + u_p = u_z + u_p$,

$$\frac{\partial}{\partial t} u_z(x,t) - \Delta u_z(x,t) = 0 \text{ in } Q, u_z(x,t) = z(x,t) \text{ on } \Sigma, u_z(x,0) = 0 \text{ on } \Omega,$$

and A is the hypersingular heat boundary integral operator, $A = D$.

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The control space

$$\mathcal{U} = H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \quad \text{vs.} \quad \mathcal{U} = L_2(\Sigma).$$

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Kunisch and Vexler [2007]; Rösch [2004]; Rösch and Wachsmuth [2011]; Winther [1978]; Arada and Raymond [2002]; ...

Optimality conditions

The reduced cost functional

$$\begin{aligned}\widetilde{J}(z) &= \frac{1}{2}\langle \mathcal{H}z + u_p - \overline{u}, \mathcal{H}z + u_p - \overline{u}\rangle_{L_2(\Omega)} + \frac{\alpha}{2}\langle Dz, z\rangle_\Sigma \\ &= \frac{1}{2}\langle \mathcal{H}^*\mathcal{H}z, z\rangle_\Sigma + \langle \mathcal{H}^*(u_p - \overline{u}), z\rangle_\Sigma + \frac{1}{2}\|u_p - \overline{u}\|^2_{L_2(\Omega)} + \frac{\alpha}{2}\langle Dz, z\rangle_\Sigma.\end{aligned}$$

Optimality condition

$$\langle \mathcal{T}_\alpha z, w - z\rangle_\Sigma \geq \langle g, w - z\rangle_\Sigma \quad \text{for all } w \in \mathcal{U}_{ad},$$

where

$$\mathcal{T}_\alpha := \alpha D + \mathcal{H}^*\mathcal{H} : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), \quad g := \mathcal{H}^*(\overline{u} - u_p) \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

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Next,

- Boundary integral representations of $\mathcal{T}_\alpha$ and g.
- Boundary element approximations.
- Error estimates.

Optimality conditions

The primal heat boundary value problem

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = 0 & \text{in } Q, \\ u(x,t) = z(x,t) & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases}$$

The adjoint heat boundary value problem

$$\begin{cases} -\partial_t p(x,t) - \Delta p(x,t) = 0 & \text{in } Q, \\ p(x,t) = 0 & \text{on } \Sigma, \\ p(x,T) = u(x,T) - \overline{u}(x) & \text{on } \Omega. \end{cases}$$

The optimality condition

$$\langle \alpha \widetilde{D} z - \frac{\partial}{\partial n} p, w - z \rangle_\Sigma \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad},$$

where

$$\widetilde{D} := \frac{1}{2}(D + \kappa_T D \kappa_T), \quad \kappa_T v(x,t) := v(x, T-t).$$

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$$\langle \alpha \widetilde{D} z - \frac{\partial}{\partial n} p, w - z \rangle_\Sigma \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad}.$$

The variational inequality reads to find $z \in \mathcal{U}_{ad}$ such that

$$\langle \mathcal{T}_\alpha z - g, w - z \rangle_\Sigma \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad}.$$

The state equation

$$\partial_t u(x,t) - \Delta u(x,t) = 0 \quad \text{in } Q, \quad u(x,t) = z(x,t) \quad \text{on } \Sigma, \quad u(x,0) = u_0(x).$$

Boundary integral equation

$$(V\omega)(x,t) = (\frac{1}{2}I + K)z(x,t) - (M_0 u_0)(x,t) \quad \text{for } (x,t) \in \Sigma,$$

to find the unknown Neumann data $\omega := \frac{\partial}{\partial n} u$, where for $(x,t) \in \Sigma$,

$$(V\omega)(x,t) = \int_0^t \int_\Gamma \mathcal{E}(x-y, t-\tau)\omega(y,\tau)\, ds_y\, d\tau,$$

and

$$(Kz)(x,t) = \int_0^t \int_\Gamma \frac{\partial}{\partial n_y}\mathcal{E}(x-y, t-\tau) z(y,\tau)\, ds_y\, d\tau.$$

Note that

$$V : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \quad \text{and } H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)\text{-elliptic.}$$

Variational inequalities

Optimality condition

$$\langle \alpha \widetilde{D} z - \frac{\partial}{\partial n} p, w - z \rangle_\Sigma \geq 0 \quad \text{for all } w \in \mathcal{U}_{ad}.$$

The variational inequality reads to find $z \in \mathcal{U}_{ad}$ such that

$$\langle \mathcal{T}_\alpha z, w - z \rangle_\Sigma \geq \langle g, w - z \rangle_\Sigma \quad \text{for all } w \in \mathcal{U}_{ad},$$

where

$$\mathcal{T}_\alpha = \alpha \widetilde{D} + \kappa_T D_1 - \kappa_T K_1' V^{-1}(\frac{1}{2} I + K) - \kappa_T (\frac{1}{2} I + K') V^{-1} K_1 \\ + \kappa_T (\frac{1}{2} I + K') V^{-1} V_1 V^{-1} (\frac{1}{2} I + K).$$

Theorem

The composed boundary integral operator $\mathcal{T}_\alpha$ is self-adjoint, bounded and $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$-elliptic, i.e.,

$$\langle \mathcal{T}_\alpha z, z \rangle_\Sigma \geq c_1^{\mathcal{T}_\alpha} \|z\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } z \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Discretization of variational inequalities

Let

$$Q_h^{1,0}(\Sigma) = S_{h_x}^1(\Gamma) \otimes T_{h_t}^0(0, T).$$

For continuous functions z_1 and z_2, define the discrete convex set

$$\mathcal{U}_h := \{w_h \in Q_h^{1,0}(\Sigma)\colon z_1(x_i, t_j) \le w_h(x_i, t_j) \le z_2(x_i, t_j) \quad \text{for all nodes } (x_i, t_j) \in \Sigma\}.$$

Then the Galerkin discretization of the variational inequality is to find $z_h \in \mathcal{U}_h$ such that

$$\langle \mathcal{T}_\alpha z_h, w_h - z_h \rangle_\Sigma \ge \langle g, w_h - z_h \rangle_\Sigma \quad \text{for all } w_h \in \mathcal{U}_h. \tag{1}$$

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The perturbed variational inequality is to find $\widetilde{z}_h \in \mathcal{U}_h$ such that

$$\langle \widetilde{\mathcal{T}}_\alpha \widetilde{z}_h, w_h - \widetilde{z}_h \rangle_\Sigma \ge \langle \widetilde{g}, w_h - \widetilde{z}_h \rangle_\Sigma \quad \text{for all } w_h \in \mathcal{U}_h, \tag{2}$$

where $\widetilde{\mathcal{T}}_\alpha$ and $\widetilde{g}$ are appropriate approximations of $\mathcal{T}_\alpha$ and g, respectively.

Abstract error estimates

Theorem

Let $z_h \in \mathcal{U}_h$ and $\widetilde{z}_h \in \mathcal{U}_h$ be the unique solutions of the variational inequalities (1) *and* (2)*, respectively. Let $\widetilde{\mathcal{T}}_\alpha$ be a bounded and $Q_h^{1,0}(\Sigma)$-elliptic approximation of $\mathcal{T}_\alpha$. Then there holds the error estimate*

$$\|z-\widetilde{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \le c_1\|z-z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}+c_2\|(\mathcal{T}_\alpha-\widetilde{\mathcal{T}}_\alpha)z\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}+c_3\|g-\widetilde{g}\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}.$$

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Theorem

Let $z_h \in \mathcal{U}_h$ be the unique solution of the variational inequality (1)*. Then there holds the error estimate*

$$\|z - z_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c\,(h_x^{s-\frac{1}{2}} + h_t^{\frac{1}{2}(s-\frac{1}{2})})\|z\|_{H^{s,\frac{s}{2}}(\Sigma)}$$

when assuming $z, z_1, z_2 \in H^{s,\frac{s}{2}}(\Sigma)$ and $\mathcal{T}_\alpha z - g \in H^{s-1,\frac{s-1}{2}}(\Sigma)$, $s \in [\frac{1}{2}, 2]$.

Error estimates

The discrete variational inequality is to find $\underline{\widetilde{z}} \in \mathbb{R}^M \leftrightarrow \widetilde{z}_h \in \mathcal{U}_h$ such that

$$(\widetilde{\mathcal{T}}_{\alpha,h}\underline{\widetilde{z}}, \underline{w} - \underline{\widetilde{z}}) \geq (\underline{\widetilde{g}}, \underline{w} - \underline{\widetilde{z}}) \quad \text{for all } \underline{w} \in \mathbb{R}^M \leftrightarrow w_h \in \mathcal{U}_h,$$

where $\widetilde{\mathcal{T}}_{\alpha,h} \in \mathbb{R}^{M\times M}$ is a symmetric and positive definite matrix, i.e.,

$$(\widetilde{\mathcal{T}}_{\alpha,h}\,\underline{z}, \underline{z}) \geq c\|z_h\|^2_{H^{1/2}(\Sigma)} \quad \text{for all } \underline{z} \in \mathbb{R}^M \leftrightarrow z_h \in Q_h^{1,0}(\Sigma).$$

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Theorem

There hold the error estimates

$$\|z - \widetilde{z}_h\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c(z, \overline{u}, u_0)h_x^{s+\frac{1}{2}},$$

and

$$\|z - \widetilde{z}_h\|_{L_2(\Sigma)} \leq c(z, \overline{u}, u_0)h_x^{s+1},$$

when assuming $z \in H^{s+1,\frac{s+1}{2}}(\Sigma)$ *for some* $s \in [0, 1]$.

Numerical results

We consider the parabolic Dirichlet boundary control problem for the domain $\Omega = B_{0.5}(O)$. The data are chosen as

$$\overline{u}(x) = (x_1^2 + x_2^2)\log(x_1^2 + x_2^2) + 4x_1x_2, \quad u_0(x) = 0, \quad \alpha = 0.1, \quad T = 0.5$$

and the box control constraints $-1 \leq z \leq 0.11$.

M	N	$\|\widetilde{z}_h - z_{ref}\|_{L_2(\Sigma)}$	eoc	$\|\widetilde{z}_h - z_{ref}\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}$	eoc
32	4	0.154249	-	1.050200	-
64	16	0.054260	1.51	0.561425	0.90
128	64	0.012887	2.07	0.212840	1.40
256	256	0.003018	2.09	0.071736	1.57
expected			2.00		1.50

Table: The results of the parabolic Dirichlet boundary control problem.

Numerical results

The final optimal solution $u(\cdot, T)$ and the target function $\overline{u}$

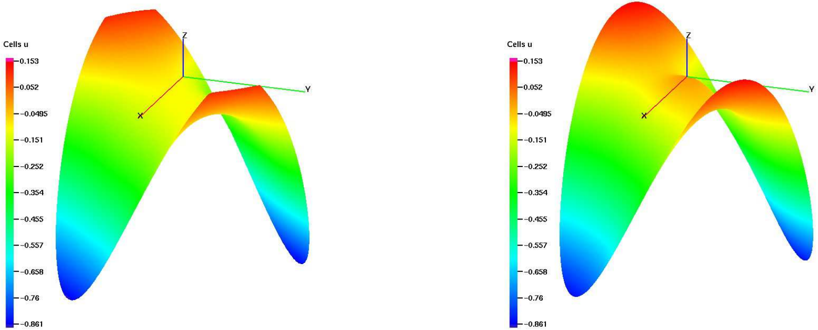

Figure: Comparison of the final optimal solution (left) and the target function (right).

Thanh Phan Xuan and O. Steinbach. Boundary element methods for parabolic boundary control problems. *J. Integral Eqn. Appl.*, 26 No. 1: 53-90 (2014).

Outline

The heat transfer coefficients

Inverse Problem I. Find a pair of functions $\{u(x,t), \sigma(\xi)\}$ such that

$$
\begin{aligned}
& u_t - \Delta u = g \quad \text{in } Q, \\
& u(x,0) = a(x), \quad x \in \overline{\Omega}, \\
& \frac{\partial u}{\partial n} + \sigma(\xi) u = b(\xi, t), \quad (\xi, t) \in \Sigma, \\
& \ell_1(u) = \chi_1(\xi), \quad \xi \in \Gamma.
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The observation operator ℓ_1 has one of the following forms:

$$\ell_1(u) = u(\xi, T_1), \quad \xi \in \Gamma,$$

where T_1 is a fixed known time in $(0, T]$, or

$$\ell_1(u) = \int_0^T \omega(t) u(\xi,t) dt, \quad \xi \in \Gamma,$$

with ω being a given function in $L^1(0,T)$.

The heat transfer coefficients

Inverse Problem II. Find a pair of functions $\{u(x,t), \sigma(t)\}$ such that

$$
\begin{aligned}
& u_t - \Delta u = g \quad \text{in } Q, \\
& u(x,0) = a(x), \quad x \in \overline{\Omega}, \\
& \frac{\partial u}{\partial n} + \sigma(t) u = b(\xi, t), \quad (\xi, t) \in \Sigma|, \\
& \ell_2(u) = \chi_2(t), \quad t \in [0, T].
\end{aligned}
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\end{aligned}
$$

The observation operator $\ell_2(u)$ has one of the following forms:

$$\ell_2(u) = u(\xi_0, t), \quad t \in [0,T],$$

where ξ_0 is fixed known point in Γ, or

$$\ell_2(u) = \int_\Gamma \nu(\xi)u(\xi,t)d\xi, \quad t \in [0,T],$$

with $\nu(\xi)$ being a given function in $L^1(\Gamma)$.

The heat transfer coefficients

With the assumptions that Ω is simply-connected and its boundary $\Gamma \in C^{1+\beta}$ with $\beta > 0$, $g \in C^{\beta,0}(\overline{Q}), a \in C^1(\overline{\Omega}), b \in C(\overline{\Sigma})$, Kostin and Prilepko [1996] proved the following results.

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Theorem

Suppose that $a = 0$ in Ω, $0 \leq g(x,t)$ in Q, $0 \leq b(\xi,t)$ on S, $0 < \chi_1(\xi)$ on Γ, and the functions g and b are monotonically non-decreasing with respect to t. Then the solution $(u(x,t), \sigma(\xi)) \in C^{2,1}(Q) \times C(\Gamma)$ with $\sigma(\xi) \geq 0$ on Γ, to the inverse problem I is unique.

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Theorem

If $|\chi_2(t)| > 0$ on $[0,T]$, then the solution $(u(x,t), \sigma(t)) \in C^{2,1}(Q) \times C[0,T]$ to the inverse problem II is unique.

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- Numerical results

Direct problem

Consider the direct problem

$$u_t - \Delta u = g \quad \text{in } Q, \tag{3}$$

$$u(x,0) = a(x), \quad x \in \Omega, \tag{4}$$

$$\frac{\partial u}{\partial n} + \sigma(\xi,t)u = b(\xi,t), \quad (\xi,t) \in \Sigma, \tag{5}$$

with

$$g \in L^2(Q), a \in L^2(\Omega), \sigma \in L^\infty(\Sigma), \sigma \geq 0, b \in L^2(\Sigma), \tag{6}$$

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$$\frac{\partial u}{\partial n} + \sigma(\xi,t)u = b(\xi,t), \quad (\xi,t) \in \Sigma, \tag{5}$$

with

$$g \in L^2(Q), a \in L^2(\Omega), \sigma \in L^\infty(\Sigma), \sigma \geq 0, b \in L^2(\Sigma), \tag{6}$$

Define

$$W(0,T) = \{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2(0,T;(H^1(\Omega))')\},$$

equipped with the norm

$$\|u\|^2_{W(0,T)} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(0,T;(H^1(\Omega))')}.$$

Direct problem

Definition

A function $u \in W(0, T)$ is called a weak solution to the direct problem (3)–(5), if

$$\int_Q (u_t\eta + \nabla u \cdot \nabla \eta)dxdt + \int_\Sigma \sigma u\eta d\xi dt = \int_Q g\eta dxdt + \int_\Sigma b\eta d\xi dt \qquad (7)$$

for all $\eta \in L^2(0, T; H^1(\Omega))$, and $u(\cdot, 0) = a$.

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Theorem

If conditions (6) are satisfied, then there exists a unique weak solution in $W(0,T)$ of the direct problem (3)–(5). Moreover, there exists a constant $c_d > 0$ independent of g, b and a such that

$$\|u\|_{W(0,T)} \leq c_d(\|g\|_{L^2(Q)} + \|b\|_{L^2(\Sigma)} + \|a\|_{L^2(\Omega)}).$$

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Boundary integral equation methods, Lax-Milgram theorem.

Variational method for the inverse problem I

Define the admissible set

$$\mathcal{A} = \{\sigma \in L^\infty(\Gamma) : 0 \le \overline{\sigma}_1 \le \sigma(\xi) \le \overline{\sigma}_2 \text{ for almost every } \xi \in \Gamma\}$$

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Suppose that χ_1 is approximately given by $\chi_{1\epsilon}$ such that

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Minimize the functional

$$\begin{aligned} J_\alpha(\sigma) &= \frac{1}{2}\|\ell_1(u(\sigma)) - \chi_{1\epsilon}\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|\sigma\|^2_{L^2(\Sigma)} \\ &= \frac{1}{2}\int_\Gamma \Big(\int_0^T \omega(t)u(\xi,t;\sigma)dt - \chi_{1\epsilon}(\xi)\Big)^2 d\xi + \frac{\alpha}{2}\|\sigma\|^2_{L^2(\Sigma)} \end{aligned}$$

over the admissible set $\mathcal{A}$.

Variational method for the inverse problem I

Lemma

The mapping $\sigma \mapsto u(\sigma)$ *is Lipschitz continuous from* $\mathcal{A}$ *to* $W(0,T)$*, i.e., for any* $\sigma, \sigma + \delta\sigma \in \mathcal{A}$*, there holds*

$$\|u(\sigma + \delta\sigma) - u(\sigma)\|_{W(0,T)} \leq c\|\delta\sigma\|_{L^\infty(\Gamma)}.$$

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Lemma

The mapping $\sigma \mapsto u(\sigma)$ *from* $\mathcal{A}$ *to* $W(0,T)$ *is Fréchet differentiable in the sense that for any* $\delta\sigma \in L^\infty(\Gamma)$ *such that* $\sigma + \delta\sigma \in \mathcal{A}$ *there exists a bounded linear operator* $\mathcal{U}$ *from* $\mathcal{A}$ *to* $W(0,T)$ *such that*

$$\lim_{\|\delta\sigma\|_{L^\infty(\Gamma)} \to 0} \frac{\|u(\sigma + \delta\sigma) - u(\sigma) - \mathcal{U}\delta\sigma\|_{W(0,T)}}{\|\delta\sigma\|_{L^\infty(\Gamma)}} = 0.$$

Variational method for the inverse problem I

The gradient of the functional J_α

We introduce the adjoint problem

$$
\begin{aligned}
-\psi_t - \Delta\psi &= 0 \text{ in } Q,\\
\psi(x, T) &= 0, \quad x \in \Omega,\\
\frac{\partial \psi}{\partial n} + \sigma(\xi)\psi &= -\omega(t)\Big(\int_0^T \omega(t)u(\xi, t; \sigma)dt - \chi_{1\epsilon}(\xi)\Big), \quad (\xi, t) \in \Sigma.
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Theorem

The functional J_α is Fréchet differentiable and its gradient is

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J'_\alpha(\sigma) = \int_0^T u(\xi,t;\sigma)\psi(\xi,t)dt + \alpha\sigma(\xi).
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Optimality system

The direct problem

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Optimality system

Optimality system

- Nonlinear conjugate gradient methods

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7. Update $\sigma_{k+1} = \sigma_k + \gamma_k d_k$.

Outline

Numerical results for the inverse problem I

Let $\Omega = (0,1) \times (0,1)$, $T = 1$. For the temperature we take the exact solution be given by

$$u(x,t) = x_1^3 + x_2^3 + 2x_1^2 x_2 \sin t + 1,$$

and then

$$a(x) = x_1^3 + x_2^3 + 1, \quad g(x,t) = 2x_1^2 x_2 \cos t - 6x_1 - 6x_2 - 4x_2 \sin t.$$

Prescribing σ, we can take b given by

$$b(\xi,t) := \frac{\partial u}{\partial n} + \sigma u, \quad (\xi,t) \in \Sigma.$$

Numerical experiments

We test the following examples

$$\textbf{Example 1} \quad \sigma(\xi) = \xi_1 + \xi_2.$$

$$\textbf{Example 2} \quad \sigma(\xi) = \begin{cases} -|\xi_1 - \frac{1}{2}| + \frac{1}{2} & \text{if} \quad \xi_2 \in \{0,1\}, \xi_1 \in (0,1), \\ -|\xi_2 - \frac{1}{2}| + \frac{1}{2} & \text{if} \quad \xi_1 \in \{0,1\}, \xi_2 \in (0,1), \end{cases}.$$

$$\textbf{Example 3} \quad \sigma(\xi) = \begin{cases} 2 & \text{if } \xi_2 = 0 \text{ or } \xi_2 = 1, \\ 1 & \text{otherwise} \end{cases}$$

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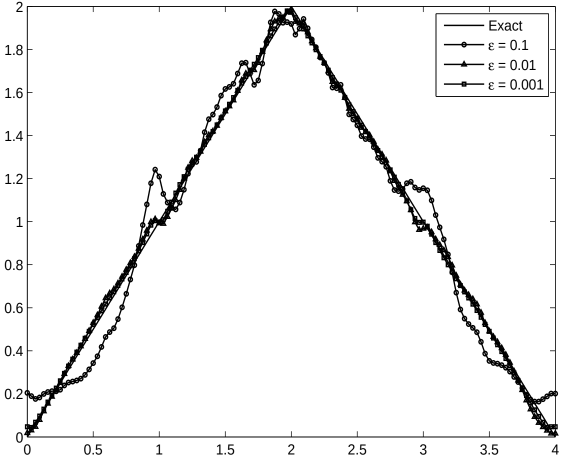
Example 3 $$\sigma(\xi) = \begin{cases} 2 & \text{if } \xi_2 = 0 \text{ or } \xi_2 = 1, \\ 1 & \text{otherwise} \end{cases}$$

Boundary elements discretization: Take $M = 128$ boundary elements and $N = 64$ time steps.

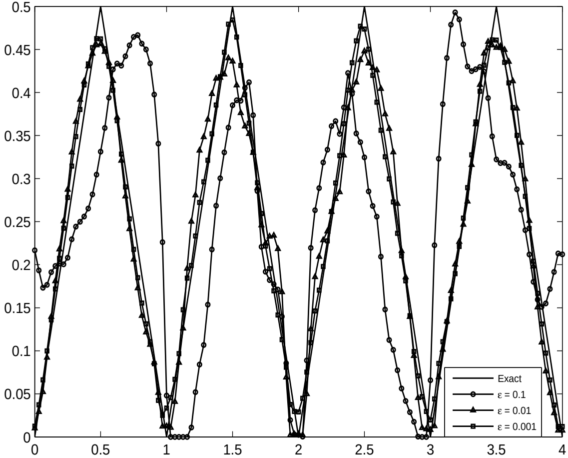
Numerical experiments

Example 1 Integral observation



Numerical experiments

Example 2 Integral observation



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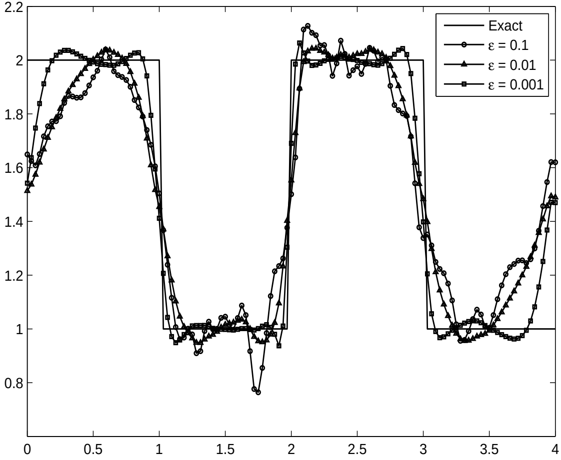


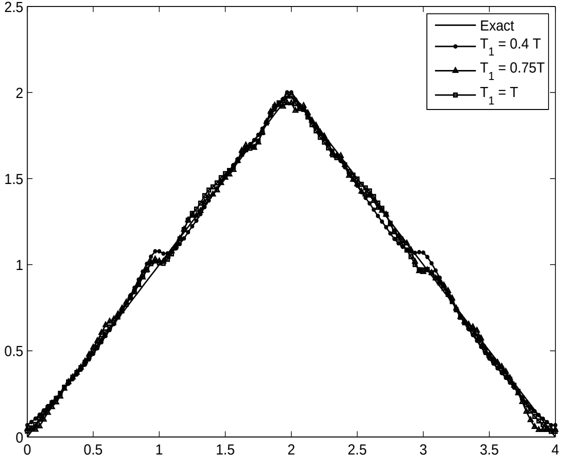
The results of the examples 1-3

Example	ϵ	n^*	$\|\sigma - \sigma_h\|_{L^2(\Gamma)}$	$J_0(\sigma_h)$
1	10^{-3}	36	0.0274	4.2E-6
1	10^{-2}	27	0.0487	9.5E-5
1	10^{-1}	8	0.1917	7.8E-3
2	10^{-3}	58	0.0270	8.3E-6
2	10^{-2}	34	0.0627	1.5E-4
2	10^{-1}	16	0.2774	8.2E-3
3	10^{-3}	40	0.2487	7.4E-5
3	10^{-2}	14	0.3905	9.8E-4
3	10^{-1}	12	0.4243	5.4E-3

Table: The stopping iteration numbers n^*, the errors $\|\sigma - \sigma_h\|_{L^2(\Gamma)}$ and the values of $J_0(\sigma_h)$ for examples 1–3 of the inverse problem I with the integral observation with $\omega(t) = t^2 + 1$, perturbed by $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ noise.

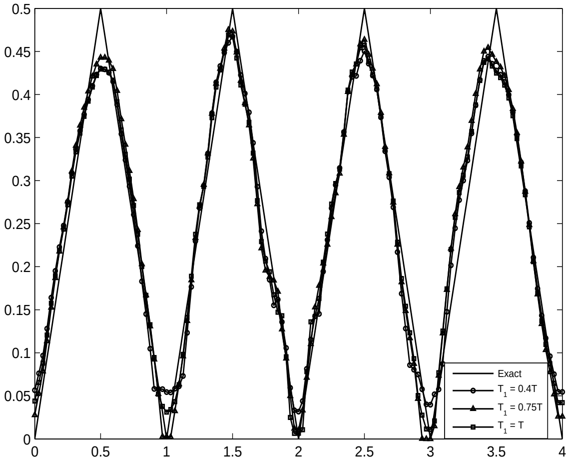
Numerical experiments

Example 1 Point-integral observation



Numerical experiments

Example 2 Point-integral observation



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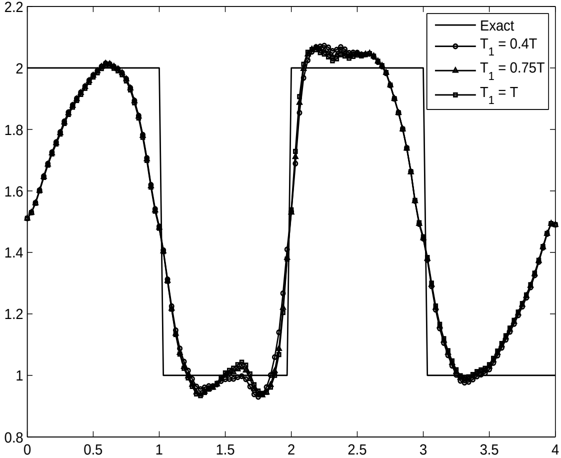
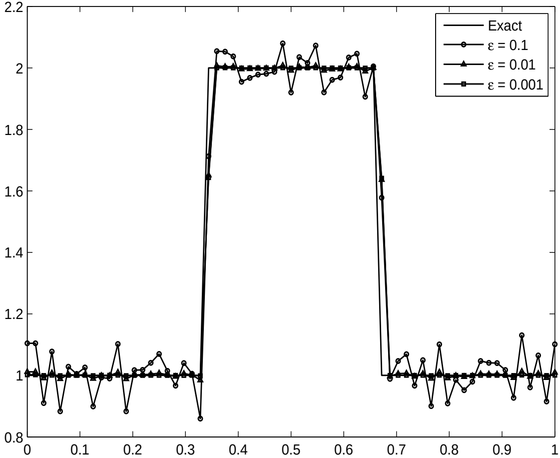


The inverse problem II

Example 4 Point-integral observation



Nonlinear heat transfer laws

Consider the inverse problem of determining the function $g(\cdot,\cdot)$

$$u_t - \Delta u = 0 \quad \text{in } Q, \tag{8}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{9}$$

$$\frac{\partial u}{\partial n} = g(u,f) \quad \text{on } \Sigma \tag{10}$$

and

1) $u|_{\Sigma_0} = h(\xi,t), \quad (\xi,t) \in \Sigma_0$ (Boundary observations),
2) $\int_\Gamma \omega(\xi)u(\xi,t)d\xi = h(t), \quad t \in (0,T]$ (Boundary integral observations).

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1) $g(u,f) = c(f-u)$ Linear boundary condition - Robin boundary condition.

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Example

1) $g(u,f) = c(f-u)$ Linear boundary condition - Robin boundary condition.
2) $g(u,f) = f^4 - u^4$ Stefan-Boltzmann radiation condition.

Numerical results

Linear case

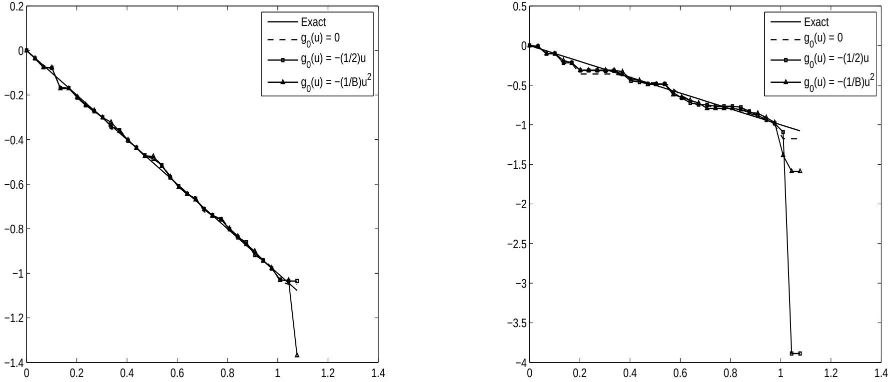

Figure: The exact linear function $g(u) = -u$ in comparison with the numerical solutions for $\delta = 0.001$ noise (left) and for $\delta = 0.01$ noise (right).

Numerical results

Nonlinear case

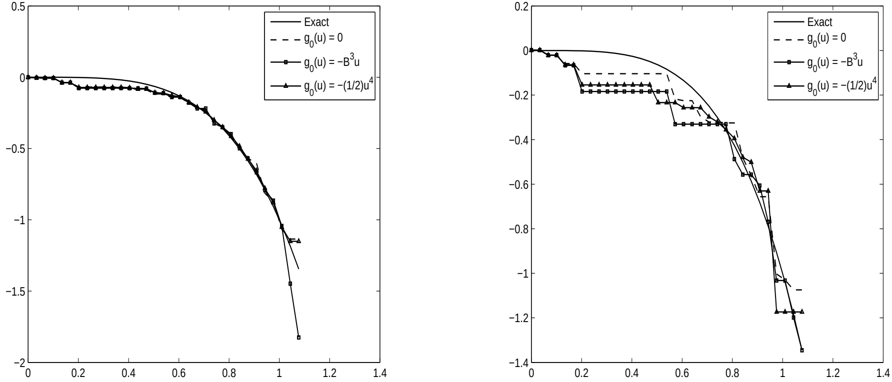

Figure: The exact nonlinear function $g(u) = -u^4$ in comparison with the numerical solutions for $\delta = 0.001$ noise (left) and for $\delta = 0.01$ noise (right).

References

Dinh Nho Hao, Phan Xuan Thanh and Daniel Lesnic, Determination of the heat transfer coefficient in transient heat conduction, *Inverse Problems*, 29(2013) 095020, 21pp.

Dinh Nho Hao, Bui Viet Huong, Phan Xuan Thanh and Daniel Lesnic, Identification of nonlinear heat transfer laws from boundary observations, *Applicable Analysis* (DOI: 10.1080/00036811.2014.948425).

Thanks for your attention!