Version 2024 FACULTY OF MATHEMATICS AND INFORMATICS, HUST

EXERCISES ON DIFFERENTIAL EQUATIONS AND SERIES – MI1046

Chapter 1: INFINITE SERIES

1.1. INFINITE SERIES OF NUMBERS

Ex. 1. In the following infinite series, which one is convergent? Then, determine its sum if any.

a) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$

b) $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$

c) $\sum\limits_{n=1}^{\infty} \sin \dfrac{n}{n+1}$

d) $\sum\limits_{n=1}^{\infty} \ln\left(1+\dfrac{1}{n}\right)$

e) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2^n}+\dfrac{(-1)^{n+1}}{3^n}\right)$

f) $\sum\limits_{n=1}^{\infty} \dfrac{n}{(2n-1)^2(2n+1)^2}$

Ex. 2 (Series of positive constants-Comparison Test/Quotient Test). Check the convergence/divergence of the following series of positive constants:

(1) $\sum\limits_{n=1}^{\infty} \dfrac{2n+3}{4n+5}$

(2) $\sum\limits_{n=1}^{\infty} \dfrac{n}{10n^2+1}$

(3) $\sum\limits_{n=2}^{\infty} \dfrac{n}{\sqrt{(n-1)(n+2)}}$

(4) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n^{\frac{3}{4}}}$

(5) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n+1}{n+2}\right)^n$

(6) $\sum\limits_{n=1}^{\infty} \sin \dfrac{1}{n^2}$

(7) $\sum\limits_{n=1}^{\infty} (\sqrt[n]{e}-1)$

(8) $\sum\limits_{n=2}^{\infty} \dfrac{2}{\ln n}$

(9) $\sum\limits_{n=2}^{\infty} \dfrac{\ln n}{n^2}$

(10) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{n}-\sin\dfrac{1}{n}\right)$

(11) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{n}-\ln\dfrac{n+1}{n}\right)$

(12) $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}\ln\dfrac{n+1}{n-1}$

(13) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}\left(\dfrac{n+1}{n}\right)^n$

(14) $\sum\limits_{n=1}^{\infty} n\sin^2\dfrac{\pi}{2\sqrt{n}}$

(15) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}\left(2^{\frac{1}{n}}-1\right)$

(16) $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{3^n+3n}$

(17) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+2^n}{3^n+2^n}$

(18) $\sum\limits_{n=1}^{\infty} n\left(e^{\frac{1}{n}}-1\right)^2$

(19) $\sum\limits_{n=1}^{\infty} \dfrac{\left(2+\cos\dfrac{n\pi}{2}\right)\sqrt{n}}{\sqrt[5]{n^7+5}}$

(20) $\sum\limits_{n=1}^{\infty} \dfrac{\arctan(2n^2+3)}{2n^2+3}$

(21) $\sum\limits_{n=2}^{\infty} \ln\left(1+\sqrt{n+2}-\sqrt{n-1}\right)$

Ex. 3 (Series of positive constants-D'Alembert Test/Cauchy Test/Integral Test). In the following series of positive constants, which one is convergent/divergent?

(1) $\sum\limits_{n=1}^{\infty} \dfrac{n^{10}}{2^n}$

(2) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+5}{2^n}$

(3) $\sum\limits_{n=1}^{\infty} \dfrac{(3n+1)!}{n^2 8^n}$

(4) $\sum\limits_{n=1}^{\infty} \dfrac{1}{5^n}\left(1-\dfrac{1}{n}\right)^{n^2}$

(5) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n+1}{n+2}\right)^{n^2}$

(6) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n-1}{n+1}\right)^{n(n-1)}$

(7) $\sum\limits_{n=1}^{\infty} \left(\dfrac{2n+3}{2n+1}\right)^{n(n-1)}$

(8) $\sum\limits_{n=1}^{\infty} \dfrac{(n+1)^{n^2}}{n^{n^2}3^n}$

(9) $\sum\limits_{n=1}^{\infty} \sqrt{n}\left(\dfrac{n}{4n-3}\right)^n$

(10) $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ (11) $\sum\limits_{n=2}^{\infty} \frac{1}{n\ln^3 n}$ (12) $\sum\limits_{n=3}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}$

Ex. 4 (**Alternating series-Leibniz Test**)**.** In the following alternating series, which one is convergent/divergent?

(1) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ (4) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ (7) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{\ln^2 n}{n}$

(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ (5) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \ln \frac{n+1}{n}$ (8) $\sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}}{n+\cos n}$

(3) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+100}$ (6) $\sum\limits_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{\sqrt{n}}$ (9) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2+1}$

Ex. 5 (**Series of sign-changing terms**-Absolute/conditional convergence)**.** In the following alternating series, which one is absolutely convergent/conditionally convergent/divergent?

(1) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$ (3) $\sum\limits_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n$ (5) $\sum\limits_{n=3}^{\infty} \frac{(-1)^n + 2\cos n}{n(\ln n)^{\frac{3}{2}}}$

(2) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$ (4) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$ (6) $\sum\limits_{n=1}^{\infty} (-1)^n \arcsin \frac{1}{n}$

1.2. SERIES OF REAL FUNCTIONS

Ex. 6. Find the domain of convergence of the following series of functions:

(1) $\sum\limits_{n=1}^{\infty} \frac{x}{(x^2+1)^n}$ (4) $\sum\limits_{n=1}^{\infty} \frac{x^n}{x^{2n}+1}$ (7) $\sum\limits_{n=1}^{\infty} \left(\frac{x(x+n)}{n}\right)^n$

(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^x}$ (5) $\sum\limits_{n=1}^{\infty} \frac{n^x+(-1)^n}{n}$ (8) $\sum\limits_{n=1}^{\infty} ne^{-nx}$

(3) $\sum\limits_{n=1}^{\infty} \frac{1}{x^n+1}$ (6) $\sum\limits_{n=1}^{\infty} \left(x+\frac{1}{n}\right)^n$ (9) $\sum\limits_{n=1}^{\infty} \frac{\ln^n\left(x+\frac{1}{n}\right)}{\sqrt{x-e}}$

Ex. 7 (**Weierstrass Test/Cauchy Test**)**.** Check the uniform convergence of the following series of functions in the given intervals:

(1) $\sum\limits_{n=1}^{\infty} \frac{x^n}{(x^2+4)^n},\ x \in \mathbb{R};$ (4) $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^2+x^2},\ x \in \mathbb{R};$

(2) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}\left(\frac{3x+1}{x+2}\right)^n,\ x \in [-1,1];$ (5) $\sum\limits_{n=1}^{\infty} x^2 e^{-nx},\ x \in [0,\infty);$

(3) $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}\sqrt{1+nx}},\ x \in [0,\infty);$ (6) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{x^2+(n+1)^2}},\ x \in \mathbb{R}.$

Ex. 8. Determine the domain of convergence of the **power series** below.

(1) $\sum\limits_{n=1}^{\infty} \dfrac{(n+2)x^n}{n^2+1}$
(2) $\sum\limits_{n=1}^{\infty} \left(\dfrac{n+1}{2n+3}\right)^n x^n$
(3) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n!}$
(4) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{2^n+3^n}$
(5) $\sum\limits_{n=1}^{\infty} \dfrac{3^n+(-2)^n}{n}(x+1)^n$
(6) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^{2n+3}}{3^{2n}(2n+3)}$
(7) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^{2n-3}}{3^{2n}\sqrt{2n+3}}$
(8) $\sum\limits_{n=1}^{\infty} \dfrac{(x+2)^{3n}}{\sqrt[3]{n+1}}$
(9) $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2+n+1}\left(\dfrac{2x-1}{x+1}\right)^n$
(10) $\sum\limits_{n=1}^{\infty} \dfrac{\ln^n x}{n^2}$
(11) $\sum\limits_{n=1}^{\infty} \dfrac{n(n+1)}{\sqrt{5n^3+n^2}}(x-2)^n$
(12) $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{(n^2+1)^2}(\sin x)^n$

Ex. 9. Calculate the sum of the following (function/constant) series:

(1) $\sum\limits_{n=1}^{\infty} nx^n,\ x \in (-1,1)$;
(2) $\sum\limits_{n=1}^{\infty} (n^2+3n-1)x^n,\ x \in (-1,1)$;
(3) $\sum\limits_{n=0}^{\infty} \dfrac{x^{2n+5}}{3^{2n}(2n+1)},\ x \in (-3,3)$;
(4) $\sum\limits_{n=0}^{\infty} \dfrac{x^{2n+2}}{(2n+1)(2n+2)},\ x \in (-1,1)$.

Ex. 10. Represent the following functions as a **MacLaurin series**:

(1) $y = \dfrac{2x+4}{x^2-3x+2}$
(2) $y = x\sin^2 x$
(3) $y = \dfrac{1}{\sqrt{4-x^2}}$
(4) $y = \dfrac{1}{x^2+x+1}$
(5) $y = \ln(1+x-2x^2)$
(6) $y = \arcsin x$
(7) $y = \dfrac{x^3+x+1}{x^2-4x+3}$
(8) $y = (x+1)e^{2x}$
(9) $y = \sin 3x + x\cos 3x$.

Ex. 11. Expand the following functions as a **Taylor series** about the given points x_0 (in a neighborhood of x_0):

(1) $f(x) = \dfrac{1}{2x+3},\ x_0 = 4$;
(2) $f(x) = \sin\dfrac{\pi x}{3},\ x_0 = 1$;
(3) $f(x) = \sqrt{x},\ x_0 = 4$;
(4) $f(x) = \ln(x^2+2x+3),\ x_0 = -1$;
(5) $f(x) = \dfrac{x^3-3x^2+4x-1}{x^2-2x+2},\ x_0 = 1$;
(6) $f(x) = \dfrac{1}{x^2+3x+2},\ x_0 = -4$.

Ex. 12. Represent the below functions as a **Fourier series** of the period $2T = 2\pi$.

(1) $y = x$, where $x \in (-\pi,\pi)$;
(2) $y = x^2$, where $x \in (-\pi,\pi)$.

Ex. 13. Expand the below functions as a **Fourier series** of the period $2T = 2$.

(1) $f(x) = x$, where $x \in (-1,1)$;
(2) $f(x) = |x|$, where $x \in (-1,1)$.

Ex. 14. Plot the graph, find their corresponding Fourier series, and determine the sum of the Fourier series at the points of discontinuity of the following functions:

(1) $f(x) = \begin{cases} 3 & \text{if } 0 < x < 2 \\ -3 & \text{if } 2 < x < 4 \end{cases}$ with period 4;

(2) $f(x) = \begin{cases} 2-x & \text{if } 0 < x < 4 \\ x-6 & \text{if } 4 < x < 8 \end{cases}$ with period 8.

Ex. 15. Find the corresponding **Fourier series** of the following functions:

a) $f(x) = \begin{cases} 1 & \text{if } \ 0 \leqslant x \leqslant \dfrac{\pi}{2} \\ 0 & \text{if } \ \dfrac{\pi}{2} < x \leqslant \pi; \end{cases}$

b) $f(x) = \begin{cases} 0 & \text{if } \ -2 < x \leqslant 0 \\ \dfrac{x}{2} & \text{if } \ 0 < x < 2. \end{cases}$

Chapter 2: DIFFERENTIAL EQUATIONS

2.1. FIRST-ORDER DIFFERENTIAL EQUATIONS

Ex. 16. Solve the first-order differential equations below.

1) **Separable equations:**

a) $y' = x^2 y$;
b) $2y(x^2+4)dy = (y^2+1)dx$;
c) $y' + e^{y+x} = 0$;
d) $1 + x + xy'y = 0$;
e) $y' = \cos^2 x \cos^2(2y)$;
f) $y^2\sqrt{1-x^2}dy = \arcsin x dx, y(0) = 0$.

2) **Homogeneous equations:**

a) $y' = \frac{y}{x} + \frac{x}{y} + 1$;
b) $xy' = x\sin\frac{y}{x} + y$;
c) $2y' + \left(\frac{y}{x}\right)^2 = -1$;
d) $(x+2y)dx - xdy = 0$;
e) $xy' = y + 2x^3\sin^2\frac{y}{x}, y(1) = \frac{\pi}{2}$;
f) $y' = \frac{y^2}{x^2} - \frac{y}{x} + 1, y(1) = 2$.

3) **Linear equations:**

a) $y' - \frac{4}{x}y = 4x^7$;
b) $xy' + y = \sqrt{x}$;
c) $y' = x - y$;
d) $xy' - y = x^2\cos x, y(\pi) = \pi$;
e) $(x^2+1)y' + 2xy = e^x$;
f) $(2xy+3)dy - y^2dx = 0$.

4) **Bernoulli equations:**

a) $y' + \frac{y}{x} = x^2y^4, y(1) = 2$;
b) $y' + \frac{2}{x}y = \frac{y^3}{x^2}$;
c) $xy' + y = -xy^2$;
d) $y' + xy = \frac{xe^{-2x^2}}{y}$;
e) $2xy' + y - \frac{x^2}{y^3}$;
f) $ydx + (x + x^2y^2)dy = 0$.

5) **Exact equations/Integrating factor**

a) $(x^2+y)dx = (2y-x)dy$;
b) $(2xy+3)dy = -y^2dx$;
c) $e^ydx = (xe^y - 2y)dy$;
d) $2x\cos^2 y dx + (2y - x^2\sin 2y)dy = 0$;
e) $(2xy^2 - 3y^3)dx = (3xy^2 - y)dy$;
f) $(3xe^y + 2y)dx + (x^2e^y + x)dy = 0$.

Ex. 17. (**Applications: RL–circuit**) An RL-circuit consists of a 100 Ohms resistor and a 2.5 Henries inductor connected in series to an electromotive force of $110\cos 314t$. Find the current at any time t.

2.2. SECOND-ORDER DIFFERENTIAL EQUATIONS

Ex. 18 (**Linear equations with constant coefficients**.). Solve the following second-order differential equations:

(1) $y'' - 3y' + 2y = 0$
(2) $y'' - 2y' + y = 0, y(1) = 2, y'(1) = -2$
(3) $y'' + 2y' + 5y = 0, y(0) = 1, y'(0) = 3$
(4) $y'' - 4y' + 3y = (15x + 37)e^{-2x}$
(5) $y'' - y = 4(x + 1)e^x$
(6) $y'' - 2y' + y = (12x + 4)e^x$
(7) $y'' - y' - 2y = xe^x \cos x$
(8) $y'' + 2y' + 10y = xe^{-x} \cos 3x$
(9) $y'' + 2y' + 2y = 8 \cos x - \sin x$
(10) $y'' + y = 2 \cos x \cos 2x$
(11) $y'' + y' - 2y = x + \sin 2x$
(12) $y'' + 3y' - 4y = 200 \sin^2 x$
(13) $y'' - 3y' + 2y = e^x + \sin x$
(14) $y'' + 4y = e^{3x} + x \sin 2x$.

Ex. 19 (**Liouville's formula**.). Solve the following second-order differential equations:

(1) $(x - 1)^2 y'' + 4(x - 1)y' + 2y = 0$, given a particular solution $y_1 = \dfrac{1}{1 - x}$;
(2) $xy'' + 2y' + xy = 0$, given a particular solution $y_1 = \dfrac{\sin x}{x}$;
(3) $y'' - \dfrac{2xy'}{x^2 + 1} + \dfrac{2y}{x^2 + 1} = 0$ given a particular solution $y_1 = x$;
(4) $x^2 y'' + xy' + \left(x^2 - \dfrac{1}{4}\right) y = 0$ with $x > 0$, given a particular solution $y_1 = \dfrac{\cos x}{\sqrt{x}}$.

Ex. 20 (**Method of variation of parameters**.). Solve the following second-order differential equations:

(1) $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$;
(2) $y'' + 2y' + y = \dfrac{e^{-x}}{x^2 + 9}$;
(3) $y'' + 4y = \dfrac{4}{\sin 2x}$;
(4) $y'' + 9y = \dfrac{3}{\cos^2 3x}$.

Ex. 21. (**Applications: Mass–spring system**) A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

Ex. 22. (**Applications: Mass–spring system**) A mass weighted 500 g stretches a spring 10 cm. The mass is acted on by an external force of $10 \sin \frac{t}{2}$ N and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. The mass is pulled down 3 cm from its equilibrium position and released. Formulate the initial value problem describing the motion of the mass.

2.3. LAPLACE TRANSFORM AND APPLICATIONS

Ex. 23. Find the Laplace transform of the following functions:

(1) $f(t) = t^2 + 3t$
(2) $f(t) = t - 2e^{3t}$
(3) $f(t) = 1 + \cosh 5t$
(4) $f(t) = \cos^2(2t)$
(5) $f(t) = (t + 1)^3$
(6) $f(t) = 2 \sin \left(t + \dfrac{\pi}{3}\right)$
(7) $f(t) = 2 \sin 3t \cos 5t$
(8) $f(t) = \sinh^2 3t$
(9) $f(t) = 3e^{2t} + 2 \sin^2 3t$.

Ex. 24. Determine the inverse Laplace transform of the following functions:

(1) $F(s) = \dfrac{3}{s^4}$
(2) $F(s) = \dfrac{3}{s-4}$
(3) $F(s) = \dfrac{5-3s}{s^2+9}$
(4) $F(s) = \dfrac{10s-3}{25-s^2}$
(5) $F(s) = \dfrac{1}{s^2-3s}$
(6) $F(s) = \dfrac{1}{s(s^2+4)}$
(7) $F(s) = \dfrac{1}{s(s+1)(s+2)}$
(8) $F(s) = \dfrac{3}{2s+4}$
(9) $F(s) = \dfrac{5-2s}{s^2+7s+10}$
(10) $F(s) = \dfrac{1}{s^3-5s^2}$
(11) $F(s) = \dfrac{1}{s^4-16}$
(12) $F(s) = \dfrac{s^2-2s}{s^4+5s^2+4}$.

Ex. 25. Solve the following initial-value problems for differential equations:

(1) $\begin{cases} x^{(3)} - x'' - x' + x = e^{2t} \\ x(0) = x'(0) = x''(0) = 0; \end{cases}$

(2) $\begin{cases} x^{(3)} - 6x'' + 11x' - 6x = 0 \\ x(0) = x'(0) = 0, x''(0) = 2; \end{cases}$

(3) $\begin{cases} x^{(4)} - 16x = 240\cos t \\ x(0) = x'(0) = x''(0) = x^{(3)} = 0; \end{cases}$

(4) $\begin{cases} x^{(4)} + 8x'' + 16x = 0 \\ x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1. \end{cases}$

Ex. 26. Solve the following initial-value problems for systems of differential equations:

(1) $\begin{cases} x' = 2x + y \\ y' = 6x + 3y \\ x(0) = 2, y(0) = 3; \end{cases}$

(2) $\begin{cases} x'' + x' + y' + 2x - y = 0 \\ y'' + x' + y' + 4x - 2y = 0 \\ x(0) = y(0) = 1, \\ x'(0) = y'(0) = 3; \end{cases}$

(3) $\begin{cases} x' + 2y' + x = 0 \\ x' - y' + y = 0 \\ x(0) = 1, y(0) = 3; \end{cases}$

(4) $\begin{cases} x'' + 2x - 4y = 0 \\ y'' - x + 2y = 0 \\ x(0) = y(0) = 0 \\ x'(0) = 1, y'(0) = -1. \end{cases}$

Ex. 27 (Shifting operation with respect to s**).** Find the Laplace transform/inverse Laplace transform of the following functions:

(1) $f(t) = t^4 e^{\pi t}$
(2) $f(t) = e^{-2t}\sin 3t$
(3) $f(t) = e^t \sin\left(t + \dfrac{\pi}{4}\right)$
(4) $f(t) = (e^t + t)^2$
(5) $f(t) = e^{2t}\cos 3t \sin t$
(6) $F(s) = \dfrac{1}{s^2+4s+4}$
(7) $F(s) = \dfrac{3s+5}{s^2-6s+25}$
(8) $F(s) = \dfrac{1}{s^3+1}$
(9) $F(s) = \dfrac{s^2+1}{s^3-2s^2+8s}$
(10) $F(s) = \dfrac{s^2+3}{(s^2+2s+2)^2}$
(11) $F(s) = \dfrac{s^3+2}{(s^2+1)(s^2-4s+5)}$
(12) $F(s) = \dfrac{2s+3}{(s^2-2s+2)(s^2+2s+5)}$.

Ex. 28 (**Shifting operation with respect to** t**).** Solve the following initial-value problems for differential equations:

$$(1)\ \begin{cases} x'' + x = f(t) \\ x(0) = x'(0) = 0, \end{cases} \quad \text{where } f(t) = \begin{cases} \cos t & \text{if } 0 \leqslant t < \pi \\ 0 & \text{if } t \geqslant \pi; \end{cases}$$

$$(2)\ \begin{cases} x'' + 4x = f(t) \\ x(0) = x'(0) = 0, \end{cases} \quad \text{where } f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < \pi \\ t & \text{if } t \geqslant \pi; \end{cases}$$

$$(3)\ \begin{cases} x'' + 4x' + 4x = f(t) \\ x(0) = 0, x'(0) = 1, \end{cases} \quad \text{where } f(t) = \begin{cases} t & \text{if } 0 \leqslant t < 1 \\ 0 & \text{if } t \geqslant 1; \end{cases}$$

$$(4)\ \begin{cases} x'' + 4x' + 5x = f(t) \\ x(0) = 1, x'(0) = 0, \end{cases} \quad \text{where } f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t < 2 \\ 0 & \text{if } t \geqslant 2. \end{cases}$$

Ex. 29. Find the Laplace transform/inverse Laplace transform of the following functions:

(1) $f(t) = t\cos^2 t$

(2) $f(t) = t^2 \sin kt$

(3) $f(t) = te^{2t}\sin 3t$

(4) $f(t) = (t - e^{2t})^2$

(5) $f(t) = \dfrac{\sin t}{t}$

(6) $f(t) = \dfrac{e^{2t} - 1}{t}$

(7) $f(t) = \dfrac{\cosh t - 1}{t}$

(8) $f(t) = \dfrac{1 - \cos 2t}{t}$

(9) $F(s) = \arctan \dfrac{2}{s+3}$

(10) $F(s) = \ln \dfrac{s^2+2}{s^2+5}$

(11) $F(s) = \ln \dfrac{s+3}{s-3}$

(12) $F(s) = \ln \dfrac{s^2+1}{(s+2)(s-3)}$

(13) $F(s) = \ln \left(1 + \dfrac{1}{s^2}\right)$

(14) $F(s) = \dfrac{e^{-3s}}{s}$.

Ex. 30. Solve the following initial-value problems for differential equations:

$$(1)\ \begin{cases} tx'' + (t-2)x' + x = 0 \\ x(0) = 0; \end{cases}$$

$$(2)\ \begin{cases} tx'' - (4t+1)x' + 2(2t+1)x = 0 \\ x(0) = 0; \end{cases}$$

$$(3)\ \begin{cases} tx'' + (4t-2)x' + (13t-4)x = 0 \\ x(0) = 0; \end{cases}$$

$$(4)\ \begin{cases} tx'' - tx' + x = 2 \\ x(0) = 2, x'(0) = -4. \end{cases}$$