EXERCISES ON ALGEBRA

Advanced Program Code: MI 1036

1) Midterm exam (proportion 0.3), writing, 60 minutes.
   Contents: Chapter 1 to Chapter 3 (Section 3.5, determinants of matrices)

2) Final exam (proportion 0.7), writing, 90 minutes.

Chương 1

Sets, Maps, and Complex Numbers

1.1. Sets and set operations

Exercise 1. Let

$$A = \{x \in \mathbb{R} | x^2 - 4x + 3 \leq 0\},\ B = \{x \in \mathbb{R} || x - 1| \leq 1\},\ C = \{x \in \mathbb{R} | x^2 - 5x + 6 \leq 0\}.$$

Compute $(A \cup B) \cap C$, $(A \cup B) \setminus C$ and $(A \cap B) \cup C$.

Exercise 2. Let A, B, C, D be arbitrary sets. Prove that

a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

b) $A \cup (B \setminus A) = A \cup B$.

c) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

d) $A \setminus (A \setminus B) = A \cap B$.

e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

f) $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$.

g) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

h) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

i) Is it true that $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$. If not, give a counterexample.

j) If $(A \cap C) \subset (A \cap B)$ and $(A \cup C) \subset (A \cup B)$, then $C \subset B$.

1.2. Mappings

Exercise 3. Let $f: X \to Y$ be a map. Prove that

a) $f(A \cup B) = f(A) \cup f(B)$, $\forall A, B \subset X$

b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, $\forall A, B \subset Y$

c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, $\forall A, B \subset Y$

d) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, $\forall A, B \subset Y$

e) $A \subset f^{-1}(f(A)), \forall A \subset X$,

f) $B \supset f(f^{-1}(B)), \forall B \subset Y$.

g) $f(A \cap B) \subset f(A) \cap f(B)$, $\forall A, B \subset X$. Give an example to show that $f(A \cap B) \neq f(A) \cap f(B)$.

Exercise 4. Let $f: \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (2x, 2y)$ and $A = \{(x, y) \in \mathbb{R}^2 \mid (x-4)^2 + y^2 = 4\}$. Find $f(A), f^{-1}(A)$.

Exercise 5. Which of the following maps are injective, surjective, bijective?

a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3 - 2x$,

b) $f : (-\infty, 0] \to [4, +\infty), f(x) = x^2 + 4$,

c) $f : (1, +\infty) \to (-1, +\infty), f(x) = x^2 - 2x$,

d) $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{3\}, f(x) = \dfrac{3x+1}{x-1}$,

e) $f : [4, 9] \to [21, 96], f(x) = x^2 + 2x - 3$,

f) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x - 2|x|$,

g) $f : (-1, 1) \to \mathbb{R}, f(x) = \ln \dfrac{1+x}{1-x}$,

h) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \dfrac{1}{x}$,

Exercise 6. Let X, Y, Z be sets and let $f : X \to Y, g : Y \to Z$ be maps. Prove that

a) If f and g are injective, then $g \circ f$ is injective.

b) If f and g are surjective, then $g \circ f$ is surjective.

c) If f and g are bijective, then $g \circ f$ is bijective.

d) If f is surjective and $g \circ f$ is injective, then g is injective.

e) Give an example to show that $g \circ f$ is injective, but g is not injective.

f) If g is is injective and $g \circ f$ is surjective, then f is surjective.

g) Give an example to show that $g \circ f$ is surjective but f is not surjective.

1.3. Algebraic structures

Exercise 7. Determine which of the following binary operations are associative:

(a) the operation $*$ on $\mathbb{R}$ defined by: $a * b = a + b + ab$

(b) the operation $*$ on $\mathbb{Z}$ defined by: $a * b = a - b$

(c) the operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ defined by: $(a, b) * (c, d) = (ad + bc, bd)$

Exercise 8. Decide which of the binary operations in the preceding exercise are commutative.

Exercise 9. Determine which of the following sets are groups under addition:

a) the set of rational numbers of absolute value < 1

b) the set of rational numbers with denominators equal to 1 or 2

c) the set of rational numbers with denominators equal to 1,2 or 3.

Exercise 10. Consider the set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ with the following binary operation $*$ defined as: for $a, b \in \mathbb{Z}_5$, $a * b = (a + b) \mod 5$ (the remainder of $(a + b)$ divided by 5). For example $2 * 4 = 1$. Show that $\mathbb{Z}_5$ is a group under this operation $*$.

Exercise 11. Which set of the following sets is a ring? a field?

(a) $X = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ (b) $Y = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

where the addition and multiplication are the common addition and multiplication

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$$
$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

1.4. Complex numbers

Exercise 12. Find the canonical forms of the following complex numbers.

a) $(1 + i\sqrt{3})^9$, b) $\dfrac{(1 + i)^{21}}{(1 - i)^{13}}$, c) $(2 + i\sqrt{12})^5(\sqrt{3} - i)^{11}$.

Exercise 13. Find all 8th roots of $1 - i\sqrt{3}$.

Exercise 14. Suppose $(3 + 4i)^{10} = a + bi$, with $a, b \in \mathbb{R}$. Find $a^2 + b^2$.

Exercise 15. Solve the following equations in the field of complex numbers.

a) $z^2 + z + 1 = 0$,

b) $z^2 + 2iz - 5 = 0$,

c) $z^4 - 3iz^2 + 4 = 0$,

d) $z^6 - 7z^3 - 8 = 0$,

e) $\dfrac{(z + i)^4}{(z - i)^4} = 1$,

f) $z^8(\sqrt{3} + i) = 1 - i$,

g) $\overline{z^7} = \dfrac{1}{z^3}$,

h) $z^4 = z + \overline{z}$.

Exercise 16. Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^4 + 1$. Find $f^{-1}(\{i\})$.

Exercise 17. Suppose $2 + i$ is a root of a polynomial $p(x) = x^3 - 2x^2 - 3x + a$. Find a.

Exercise 18. Suppose $1 + 2i$ is a root of a real polynomial $p(x) = x^3 - ax^2 + bx - (2a + 2)$. Find a, b.

Exercise 19. Let $\epsilon = \cos(\frac{2\pi}{7}) + i\sin(\frac{2\pi}{7})$. Show that

(a) $\epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \epsilon^5 + \epsilon^6 = -1$;

(b) $\epsilon + \epsilon^2 - \epsilon^3 + \epsilon^4 - \epsilon^5 - \epsilon^6 = i\sqrt{7}$;

Exercise 20. Let $\epsilon = \cos(\frac{2\pi}{15}) + i\sin(\frac{2\pi}{15})$. Show that

$$\epsilon + \epsilon^2 + \epsilon^4 + \epsilon^7 + \epsilon^8 + \epsilon^{11} + \epsilon^{13} + \epsilon^{14} = 1.$$

Chương 2

Matrices, System of Linear Equations

2.1-2.2. Matrix operations

Exercise 21. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ 0 & 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

Compute $A + BC, A^T B - C, A(BC), (A + 3B)(B - C)$.

Exercise 22. Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$.

a) Compute $F = A^2 - 3A$,

b) Find the matrix X satisfying $(A^2 + 5I)X = B^T(3A - A^2)$.

Exercise 23. Find the matrix X such that:

a) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + 2X = \begin{bmatrix} 1 & -2 \\ 5 & 7 \end{bmatrix}$.

b) $\frac{1}{2}X - \begin{bmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \\ 2 & -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 6 \\ -2 & -9 & 2 \\ -4 & -8 & 6 \end{bmatrix}$.

Exercise 24. Find a real 2×2 matrix $A \neq 0$ such that: a)$A^2 = 0$, b) $A^2 = -I_2$.

Exercise 25. Find two real 2×2 matrices A and B such that $(A + B)^2 \neq A^2 + 2AB + B^2$.

Exercise 26. Use the given definition to find $f(A)$: If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$, then for an $n \times n$ matrix A, $f(A)$ is defined to be $f(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_d A^d$.

a) $f(x) = x^2 - 5x + 2$, $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$; b) $f(x) = x^2 - 7x + 6$, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$;

c) $f(x) = x^3 - 2x^2 + 5x - 10$, $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$.

Exercise 27. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that $A^2 - (a+d)A + (ad - bc)I_2 = O_2$.

Exercise 28. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. Check that $A^2 - 4A - 5I_2 = O_2$, and compute A^n $(n \in \mathbb{N})$.

Exercise 29. Compute A^n, where: a) $A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$, b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$.

Exercise 30. Let A be a square matrix. Show that

a) AA^T, A^TA and $A + A^T$ are symmetric matrices.

b) $A - A^T$ is a skew-symmetric matrix.

Exercise 31. Let $A \in M_n(\mathbb{R})$ be a matrix such that $AA^T = O_n$. Show that $A = O_n$.

Exercise 32. Let $A, B \in M_n(\mathbb{R})$ be two matrices such that $AA^T + BB^T = AB^T + BA^T$. Show that $A = B$.

2.3. Linear systems of equations

Exercise 33. Solve the following systems of linear equations

a) $\begin{cases} x_1 - 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 + x_2 + x_3 &= -1 \end{cases}$

b) $\begin{cases} x_1 - 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 - 3x_2 + 2x_3 &= 4 \end{cases}$

c) $\begin{cases} 3x_1 - 5x_2 + 2x_3 + 4x_4 &= 2 \\ 7x_1 - 4x_2 + x_3 + 3x_4 &= 5 \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 &= 3 \end{cases}$

d) $\begin{cases} 3x_1 - x_2 + 3x_3 &= 1 \\ -4x_1 + 2x_2 + x_3 &= 3 \\ -2x_1 + x_2 + 4x_3 &= 4 \\ 10x_1 - 5x_2 - 6x_3 &= -10 \end{cases}$

Exercise 34. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{cases} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2. \end{cases}$$

Chương 3

Vector spaces, rank and inverse of a matrix

3.1. Vector spaces and subspaces

Exercise 35. Determine whether V is a vector space?

a) $V = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$, the operations are defined as

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z'); \quad k(x, y, z) = (|k|\, x, |k|\, y, |k|\, z)\ (k \in \mathbb{R}).$$

b) $V = \{x = (x_1, x_2) \mid x_1 > 0, x_2 > 0\} \subset \mathbb{R}^2$, the operations are defined as

$$(x_1, x_2) + (y_1, y_2) = (x_1 y_1, x_2 y_2); \quad k(x_1, x_2) = (x_1^k, x_2^k), \quad k \in \mathbb{R}.$$

Exercise 36. For each of the following subsets of $\mathbb{R}^3$, determine whether it is a subspace of $\mathbb{R}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + x_3 = 0\}$;

(b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + x_3 = 1\}$;

(c) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 + x_3 > 0\}$;

(d) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 x_3 = 0\}$;

(e) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 3x_3\}$.

Exercise 37. Let V_1, V_2 be linear subspaces of V and $V_1 + V_2 := \{x_1 + x_2 \mid x_1 \in V_1, x_2 \in V_2\}$. Prove that:

a) $V_1 \cap V_2$ is a linear subspace of V.

b) $V_1 + V_2$ is a linear subspace of V.

Exercise 38. Let V_1, V_2 be subspaces of V. Assume that

i) $\{v_1, v_2, \cdots, v_m\}$ be a set of generators (a generating set) of V_1, and

ii) $\{u_1, u_2, \cdots, u_n\}$ be a set of generators of V_2.

Prove that $\{v_1, \cdots, v_m, u_1, u_2, \cdots, u_n\}$ is a set of generators of $V_1 + V_2$.

Exercise 39. Prove that $V = V_1 \oplus V_2$ [1] if and only if each $v \in V$ has a unique representation

$$v = v_1 + v_2,\ (v_1 \in V_1, v_2 \in V_2).$$

3.2. Dimension and coordinates

Exercise 40. Write v as a linear combination of u_1, u_2 and u_3 if possible, where

$$v = (3, 0, -6),\ u_1 = (1, -1, 2),\ u_2 = (2, 4, -2),\ u_3 = (1, 2, -4).$$

Exercise 41. Express the polynomial $v = t^2 + 4t - 3$ over $\mathbb{R}$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 3$.

Exercise 42. Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$, that is, so that w belongs to $\text{span}(u, v)$.

Exercise 43. Is the vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?

Exercise 44. Determine whether the following vectors are linearly dependent or linearly independent.

a) $(1, 2, -1), (2, 1, -1), (7, -4, 1)$.

b) $(2, 3, -1), (3, -1, 5), (1, 7, -7)$.

Exercise 45. Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f, g, h \in V$ are linearly indepedent, where $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$.

Exercise 46. Let v_1, v_2 and v_3 be three linearly independent vectors in a vector space V.

a) Prove that $\{v_1 - v_2, v_2 - v_3, v_1 + v_2 + v_3\}$ is linearly independent.

b) Prove that $\{v_1 - v_2, v_2 - v_3, v_1 - 2v_2 + v_3\}$ is linearly dependent.

c) For which values of a is the set $\{v_1 - v_2, v_2 - v_3, v_1 + av_2 + v_3\}$ linearly independent?

Exercise 47. Determine whether the set S is a basis of $\mathbb{R}^3$

a) $S = \{(1, 2, 1), (1, 1, 1)\}$;

b) $S = \{(1, 1, 7), (3, 1, -3), (2, 1, 2)\}$;

c) $S = \{(1, 0, -1), (1, 2, 1), (0, -3, 2)\}$.

Exercise 48. Determine whether the set S in $\mathcal{P}_2[x]$ is a basis. (Here $\mathcal{P}_2[x]$ is the vector space of polynomials in x with real coefficients of degree ≤ 2.)

a) $S = \{1 + x, 2 + x + x^2, 3 - 2x + x^2\}$;

b) $S = \{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 3\}$.

[1] We say that V is a direct sum of V_1 and V_2 and write $V = V_1 \oplus V_2$ if $V_1 + V_2 = V, V_1 \cap V_2 = \{0\}$.

Exercise 49. Find a basis and the dimension of subspace W of $\mathbb{R}^3$.

a) $W = \{(a, a+b, a-2b) \mid a, b \in \mathbb{R}\}$; b) $W = \{(x, y, z) \mid x+y+z = 0\}$;

c) $W = \{(x, y, z) \mid x - 2y + z = 0 \text{ and } 2x - 3y + z = 0\}$.

Exercise 50. Let W be a subspace of $\mathbb{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), u_2 = (2, 3, 1, -4), u_3 = (3, 8, -3, -5).$$

a) Find a basis and the dimension of W.

b) Extend the basis of W found in part a) to a basis of the whole space $\mathbb{R}^4$.

Exercise 51. Find the coordinate vector of x relative to (with respective to) the basis B of $\mathbb{R}^m$:

a) $B = \{(1, 1), (0, -2)\},\ x = (2, -1)$.

b) $B = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\},\ x = (4, -2, 9)$.

c) $B = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\},\ x = (3, -3, 0)$.

Exercise 52. Find the coordinate vector of x relative to the basis B', where

$$B = \{(1, 1), (1, -1)\},\ B' = \{(0, 1), (1, 2)\},\quad [x]_B = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Exercise 53. Find the coordinates of $p(x) = 6 - 7x + x^2$ relative to the basis S of $\mathcal{P}_2[x]$, where

$$S = \{1 + x, 2 + x + x^2, 3 - 2x + x^2\}.$$

Exercise 54. Consider the subspaces $U = \text{span}(u_1, u_2, u_3)$ and $W = \text{span}(w_1, w_2, w_3)$ of $\mathbb{R}^3$ where

$$u_1 = (1, 1, -1), u_2 = (2, 3, -1), u_3 = (3, 1, -5), w_1 = (1, -1, -3), w_2 = (3, -2, -8), w_3 = (2, 1, -3).$$

Show that $U = W$.

Exercise 55. Let $v_1 = 1, v_2 = 1 + x, v_3 = x + x^2, v_4 = x^2 + x^3$ be vectors on $P_3[x]$.

a) Prove that $B = \{v_1, v_2, v_3, v_4\}$ is a basis of $P_3[x]$.

b) Find the coordinates of $v = 2 + 3x - x^2 + 2x^3$ with respect to this basis.

c) Find the coordinates of $v = a_0 + a_1x + a_2x^2 + a_3x^3$ with respect to this basis.

Exercise 56. Let $E = \{1, x, x^2, x^3\}$ be the standard basis of $P_3[x]$ and $B = \{1, 1 + x, (1 + x)^2, (1 + x)^3\}$.

a) Prove that B is a basis of $P_3[x]$.

b) Find the transformation matrices from E to B, and from B to E.

c) Find the coordinates of $v = 2 + 2x - x^2 + 3x^3$ with respect to the basis B.

3.3. Rank

Exercise 57. Find the rank of the following family of vectors on $P_3[x]$:

$$v_1 = 1 + x^2 + x^3, v_2 = x - x^2 + 2x^3, v_3 = 2 + x + 3x^3, v_4 = -1 + x - x^2 + 2x^3.$$

Exercise 58. Find the rank of the following matrices

a) $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & -1 & 4 & -6 \end{bmatrix}$.

3.4. Linear systems of equations revisited

Exercise 59. Find the dimension and a basis of the solution space of the homogeneous system

$$\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 - x_5 = 0 \\ x_1 - 2x_2 + 3x_3 - x_4 + 5x_5 = 0 \\ 2x_1 + x_2 + x_3 + x_4 + 3x_5 = 0 \\ 3x_1 - x_2 - 2x_3 - x_4 + x_5 = 0 \end{cases}$$

3.5. The inverse and determinant of a matrix

Exercise 60. Find the inverses of the matrices

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

b) $C = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

c) $D = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 61. Compute the following determinants

a) $A = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{vmatrix}$

c) $C = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$

b) $B = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$

d) $D = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}$.

Exercise 62. Prove that if A is a skew-symmetric (or antisymmetric) matrix of order n, where n is odd, then $\det(A) = 0$.

Exercise 63. Let A be a square matrix of order 2017. Prove that

$$\det(A - A^T)^{2017} = 2017(\det A - \det A^T).$$

Exercise 64. Let A, B be square matrices of order 2017 satisfying $AB + B^T A^T = 0$. Prove that $\det A = 0$ or $\det B = 0$.

Exercise 65. Let $A, B \in M_n(\mathbb{R})$. Suppose that $AB = BA$. Show that

a) $\det(A^2 + B^2) \geq 0$.

b) $\det(A^2 + AB + B^2) \geq 0$.

Exercise 66. Let $A, B \in M_n(\mathbb{R})$. Suppose that $A^2 + B^2 = O_n$ and $AB - BA$ is invertible. Show that n is even.

Exercise 67. Prove that if A is a real square matrix satisfying $A^3 = A + I$, then $\det A > 0$. (Hint: $A^5 = A^2 + A + I$.)

Exercise 68. Let A, B be square matrices of the same order satisfying $AB = A + B$. Prove that $AB = BA$.

Exercise 69. Let A, B be two 3×3 matrices such that $A^2 = AB + BA$. Prove that $\det(AB - BA) = 0$. (Hint: $AB - BA = A^2 - 2BA = A(A - 2B)$, then taking determinant both sides.)

Chương 4

Linear mappings and transformations

4.1-4.3. Linear mappings

Exercise 70. If $\alpha_1 = (1,-1), \alpha_2 = (2,-1), \alpha_3 = (-3,2), \beta_1 = (1,0), \beta_2 = (0,1), \beta_3 = (1,1)$, is there a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\alpha_i) = \beta_i$ for $i = 1,2,3$?

Exercise 71. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping for which $T(1,2) = (2,3)$ and $T(0,1) = (1,4)$. Find a formula for T, that is, find $T(a,b)$ for arbitrary a and b.

Exercise 72. Suppose $b, c \in \mathbb{R}$. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x,y,z) = (2x-4y+3z+b, 6x+cxyz)$ Show that T is linear if and only if $b = c = 0$.

Exercise 73. Let T be the function from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T(x_1,x_2,x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

a) Verify that T is a linear transformation.

b) Show that $(a,b,c) \in \operatorname{im}T$ if and only if $-a+b+c = 0$.

c) Find a basis of $\operatorname{im}T$.

d) Find a basis of $\ker T$.

Exercise 74. Find a basis for (a) $\ker(T)$ and (b) $\operatorname{im}(T)$, where

$$T\colon \mathbb{R}^4 \to \mathbb{R}^3,\ T(x,y,z,w) = (4x - 5y + 5z + 2w, -2x + 2y - w, -y + 5z).$$

Exercise 75. Let $T\colon V \to U$ be linear, and suppose $v_1, \ldots, v_n \in V$ have the property that their images $T(v_1), \ldots, T(v_n)$ are linearly indepedent. Show that the vectors $v_1, \ldots, v_n$ are also linearly independent.

Exercise 76. Suppose $T\colon V \to U$ be an injective linear map and $v_1, \ldots, v_m$ are linearly independent in V. Show that $T(v_1), \ldots, T(v_m)$ are linearly independent in U.

Exercise 77. Give an example of a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $\operatorname{im}T = \ker T$.

Exercise 78. Prove that there does not exist a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ such that $\operatorname{im}T = \ker T$.

Exercise 79. Suppose $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ is a linear map such that

$$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Exercise 80. Prove that there does not exist a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ such that

$$\ker T = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Exercise 81. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map defined $f(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 2x_1 + x_3)$. Find the matrix of f with respect to the standard bases.

Exercise 82. Find the matrix of T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$, where

$$T\colon \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (-x, y, x + y), \mathcal{B} = \{(1, 1), (1, -1)\}, \mathcal{B}' = \{(0, 1, 0), (0, 0, 1), (1, 0, 0)\}.$$

Exercise 83. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be be a function defined by

$$f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + x_3).$$

Find the matrix of f with respect to the basis $B = \{v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)\}$.

Exercise 84. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 2y - z)$. (a) Show that T is invertible. Find formulas for: (b) T^{-1}, (c) T^2, (d) T^{-2}.

Exercise 85. Let the function $f\colon P_2[x] \to P_4[x]$ be a map defined as: $f(p) = p + x^2 p, \forall p \in P_2[x]$.

a) Prove that f is a linear map.

b) Find the matrix of f with respect to the bases $E_1 = \{1, x, x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.

c) Find the matrix of f with respect to the bases $E_1' = \{1 + x, 2x, 1 + x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.

Exercise 86. Suppose $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ is the matrix of a linear transformation $f\colon P_2[x] \to P_2[x]$ with respect to the basis $B = \{v_1, v_2, v_3\}$, where

$$v_1 = 3x + 3x^2, v_2 = -1 + 3x + 2x^2, v_3 = 3 + 7x + 2x^2.$$

a) Find $f(v_1), f(v_2), f(v_3)$.

b) Find $f(1 + x^2)$.

Exercise 87. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

Exercise 88. Let A, B be $m \times n$ matrices. Prove that $\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$.

Chương 5

Eigenvalues and eigenvectors

5.1. Eigenvalues and eigenvectors

Exercise 89. Find the eigenvalues and a basis for each eigenspace of the following matrices:

a) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{bmatrix}$

Exercise 90. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (-2x+2y-3z, 2x+y-6z, -x-2y)$. Find all eigenvalues and a basis for each eigenspace of T.

Exercise 91. Let $f\colon P_2[x] \to P_2[x]$ be a linear transformation defined by

$$f(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find the eigenvalues and eigenvectors of f.

5.2-5.3. Properties of eigenvalues and eigenvectors, diagonalization

Exercise 92. Suppose $T\colon V \to V$ is linear with $\text{rank}T = k$. Prove that T has at most $k+1$ distinct eigenvalues.

Exercise 93. Suppose $T\colon V \to V$ is linear and there exist a nonzero vectors v and w in V such that $Tv = 3w$ and $Tw = 3v$. Prove that 3 or -3 is an eigenvalue of T.

Exercise 94. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is linear and that $-4, 5, \sqrt{6}$ are eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 7x = (-4, 5, \sqrt{6})$.

Exercise 95. Let $\lambda_1, \dots, \lambda_n$ be a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$. [Hint: Let $V = \text{span}\{e^{\lambda_1 x}, \dots, e^{\lambda_n x}\}$ and define $T\colon V \to V$ by $Tf = f'$.]

Exercise 96. Diagonalize the following matrices (if possible)

a) $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

e) $E = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

Exercise 97. Suppose that $A, B \in M_3(\mathbb{R})$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible matrix $P \in M_3(\mathbb{R})$ such that $B = P^{-1}AP$.

Exercise 98. Let $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined as

$$f(x_1, x_2, x_3) = (2x_1 - x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + 2x_3).$$

Diagonalize the transformation f.

Exercise 99. Find a basis of $\mathbb{R}^3$ such that the matrix of $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ with respect to this basis is a diagonal matrix, where

$$f(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + 2x_2 + x_3, x_1 + x_2 + 2x_3).$$

Exercise 100. Let V be the $\mathbb{R}$-vector space of all polynomials $p(x) \in \mathbb{R}[x]$ with $\deg(p) \leq 2$. Let $T \colon V \to V$ be the linear transformation given by

$$T(a + bx + cx^2) = (a + 3b + 3c) + (3a + b + 3c)x + (3a + 3b + c)x^2.$$

If possible find a basis B for V such that the matrix of T with respect to B is diagonal. (Diagonalize the transformation T.)

Exercise 101. The trace of an n-by-n square matrix A is defined to be the sum of the elements on the main diagonal,i.e., $\operatorname{tr}(A) = \sum\limits_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$. Prove that

a) The trace is a linear mapping. That is,

  i) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, ii) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$;

b) $\operatorname{tr}(A) = \operatorname{tr}(A^T)$, c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, d) $\operatorname{tr}(P^{-1}AP) = \operatorname{tr} A$.

Exercise 102. Let $A \in M_n(\mathbb{C})$ be an invertible matrix and $0 \neq \lambda \in \mathbb{C}$. Show that λ is an eigenvalue of A if and only if $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Exercise 103. Let $P(x) \in \mathbb{C}[x]$ be a polynomial and let A be a square matrix. Show that if λ is an eigenvalue of A then $P(\lambda)$ is an eigenvalue $P(A)$.

Exercise 104. Let $A \in M_n(\mathbb{C})$ and let $P \in \mathbb{C}[x]$ be a polynomial such that $P(A) = 0$. Prove that any eigenvalue λ of A satisfies $P(\lambda) = 0$.

Chương 6

Euclidean spaces, orthogonality

6.1. Inner products

Exercise 105. Verify that the following is an inner product on $\mathbb{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$\langle u \rangle v = x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2.$$

Exercise 106. Find the values of k so that the following is an inner product on $\mathbb{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$\langle u \rangle v = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + kx_2 y_2.$$

Exercise 107. Determine if each of the following is an inner product on $P_3[x]$:

a) $\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$

b) $\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) + p(3)q(3)$

c) $\langle p, q \rangle = \int\limits_{-1}^{1} p(x)q(x)dx.$

In case it is an inner product, compute $\langle p, q \rangle$, where $p = 2 - 3x + 5x^2 - x^3, q = 4 + x - 3x^2 + 2x^3$.

Exercise 108. Suppose $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Exercise 109. (a) Suppose $u, v, w \in \mathbb{R}^n$. Prove that

$$\|w - \frac{1}{2}(u+v)\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

(b) Suppose C is a subset of $\mathbb{R}^n$ with the property that $u, v \in C$ implies $\frac{1}{2}(u+v) \in C$. Let $w \in V$. Show that there is at most one $u \in C$ such that

$$\|w-u\| \leq \|w-v\| \quad \text{for all } v \in C.$$

6.2. Orthogonality

Exercise 110. Let the inner product on $P_2[x]$ be defined as $\langle p, q \rangle = \int\limits_{-1}^{1} p(x)q(x)dx$, where $p, q \in P_2[x]$.

a) Apply the Gram-Schmidt process to the basis $\{1, x, x^2\}$ to get an orthonormal basis $\mathcal{A}$.

b) Find the coordinate vector $[r]_{\mathcal{A}}$, where $r = 2 - 3x + 3x^2$.

In the following exercises (Ex 111-Ex 121), we consider $\mathbb{R}^n$ or $M_{n\times 1}(\mathbb{R})$ with the standard inner product.

Exercise 111. Find vectors $u, v \in \mathbb{R}^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$ and $(1, 2) = u + v$.

Exercise 112. Let S consist of the following vectors of $\mathbb{R}^4$:

$$u_1 = (1, 1, 1, 1), u_2 = (1, 1, -1, -1), u_3 = (1, -1, 1, -1), u_4 = (1, -1, -1, 1).$$

a) Show that S is orthogonal and a basis of $\mathbb{R}^4$.

b) Write $v = (1, 3, -5, 6)$ as a linear combination of u_1, u_2, u_3, u_4.

c) Find the coordinates of an arbitrary vector $v = (a, b, c, d)$ in $\mathbb{R}^4$ relative to the basis S.

d) Normalize S to obtain an orthonormal basis of $\mathbb{R}^4$.

Exercise 113. Use the Gram-Schmidt process to transform the basis B into an orthonormal basis.

(a) $B = \{(1, 1), (0, 1)\}$.

(b) $B = \{(1, -2, 2), (2, 2, 1), (2, -1, -2)\}$,

(c) $B = \{(1, 2, -2), (0, 1, -2), (-1, 3, 11),$

(d) $B = \{(3, 4, 0, 0), (-1, 1, 0, 0), (2, 1, 0, -1), (0, 1, 1, 0)\}$.

Exercise 114. Let $v_1 = (1, 1, 0, 0, 0)\,, v_2 = (0, 1, -1, 2, 1)\,, v_3 = (2, 3, -1, 2, 1)\,,$ and

$$V = \left\{x \in \mathbb{R}^5 \mid x \perp v_i,\ i = 1, 2, 3\right\}.$$

a) Prove that V is a subspace of $\mathbb{R}^5$. b) Find a baisis of V and $\dim V$.

Exercise 115. Let W be a the solution space of the homogeneous system of linear equations

$$x + y - z + w = 0$$
$$2x + y + z + 2w = 0.$$

(a) Find an orthonormal basis for W.

(b) Find an orthonormal basis for $W^\perp$.

(c) Find a system of linear equations for which $W^\perp$ is its solution space.

Exercise 116. Find the (orthogonal) projection of $u = (1, 3, -2, 4)$ on $v = (2, -2, 4, 5)$.

Exercise 117. Let $v_1 = (2, 2, 1)\,, v_2 = (2, 5, 4)$. Find the (orthogonal) projection of $v = (3, -2, 1)$ onto $U = \operatorname{span}(v_1, v_2)$.

6.3. Least square approximations

Exercise 118. In $\mathbb{R}^4$, let $U = \operatorname{span}\{(1, 1, 0, 0), (1, 1, 1, 2), (2, 2, 1, 2)\}$. Find $u \in U$ such that $\|u - (1, 2, 3, 4)\|$ is as small as possible.

Exercise 119. Find $a, b \in \mathbb{R}$ such that

$$(a + b - 1)^2 + (a + b - 2)^2 + (b - 3)^2 + (2b - 4)^2$$

is as small as possible.

Exercise 120. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find a column vector $\tilde{X} \in M_{3\times 1}(\mathbb{R})$ for which minimizes the function $f(X) = \|AX - B\|$ defined for all $X \in M_{3\times 1}(\mathbb{R})$.

6.4. Orthogonal diagonalization

Exercise 121. Orthogonally diagonalize of the following symmetric matrices

a) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$.

6.5. Quadratic forms

Exercise 122. Determine the definiteness of the following quadratic form on $\mathbb{R}^3$.

a) $\omega_1(x_1, x_2, x_3) = x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3,$

b) $2x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3,$

c) $\omega_2(x_1, x_2, x_3) = x_1x_2 + 4x_1x_3 + x_2x_3,$

Exercise 123. Find a such that the following quadratic forms are positive definite:

a) $5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$,

b) $2x_1^2 + x_2^2 + 3x_3^2 + 2ax_1x_2 + 2x_1x_3$,

c) $x_1^2 + x_2^2 + 5x_3^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3$.

Exercise 124. Orthogonally diagonalize of the following quadratic forms

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$,

b) $7x_1^2 - 7x_2^2 + 48x_1x_2$,

c) $7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 + 4x_2x_3$.

Exercise 125. Transform the following quadric surface to the principal axes:

$$2x^2 + 6y^2 + 14z^2 - 6xy + 2xz + 6yz + 2x - y + z = 0.$$

Exercise 126. Classify the following quadratic curves

a) $2x^2 - 4xy - y^2 + 8 = 0$,

b) $x^2 + 2xy + y^2 + 8x + y = 0$,

c) $2x^2 + 4xy + 5y^2 = 24$.

Exercise 127. Classify the following quadric surfaces

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 = 4$,

b) $2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 = 16$,

c) $2xy + 2yz + 2xz - 6x - 6y - 4z = 0$.

Exercise 128. Let $Q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$.
Find $\max\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$, $\min\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$.

Exercise 129. Let $Q(x, y, z) = 9x_1^2+7x_2^2+11x_3^2-8x_1x_2+8x_1x_3$ $(x, y, z \in \mathbb{R})$. Find $\max\limits_{Q(x_1,x_2,x_3)=16} x_1^2 + x_2^2 + x_3^2$, and $\min\limits_{Q(x_1,x_2,x_3)=16} x_1^2 + x_2^2 + x_3^2$.

Exercise 130. Is there an orthogonal matrix $A \in M_3(\mathbb{R})$ such that

$$A\begin{bmatrix}1\\3\\0\end{bmatrix} = \begin{bmatrix}3\\0\\1\end{bmatrix} \text{ and } A\begin{bmatrix}-3\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-3\\0\end{bmatrix}?$$

Exercise 131. Is there a symmetric matrix $A \in M_3(\mathbb{R})$ such that

$$A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \text{ and } A\begin{bmatrix}2\\3\\5\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}?$$

Exercise 132. Let A, B be $n \times n$ matrices on $\mathbb{R}$. Prove that:

a) All the eigenvalues of A are positive if and only if $X^T AX > 0$ for all $X \in M_{n\times 1}(\mathbb{R}) \setminus \{0\}$.

b) If all the eigenvalues of A and B are positive, then so are the eigenvalues of $A + B$.