Hanoi University of Science and Technology
School of Applied Mathematics and Informatics
Exercises: Calculus 2
Course ID: MI 1026

Chapter 1. Vectors and the geometry of space

Exercise 1. Determine whether the given vectors are orthogonal, parallel, or neither

(a) $\vec{a} = (-5; 3; 7), \vec{b} = (6; -8; 2)$.

(b) $\vec{a} = (4; 6), \vec{b} = (-3; 2)$.

(c) $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}, b = 3\vec{i} + 4\vec{j} - \vec{k}$.

(d) $\vec{u} = (a, b, c), \vec{v} = (-b, a, 0)$.

Exercise 2. For what values of b are the vectors $(-6; b; 2)$ and $(b; 2b; b)$ orthogonal?

Exercise 3. Find two unit vectors that make an angle of 30^0 with $v = (3; 4)$.

Exercise 4. Find the angle between a diagonal of a cube and one of its edges.

Exercise 5. Find an unit vector that parallel with $8\vec{i} - \vec{j} + 4\vec{k}$.

Exercise 6. Find area of triangle ABC, where $A(2; 8; 12)$, $B(4; 5; 8)$, $C(1; 4; 10)$.

Exercise 7. Find the altitude AH of triangle ABC, where $A(1; 6; 4)$, $B(2; 5; 8)$ and $C(-1; 4; 0)$.

Exercise 8. Prove that $\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z}) \cdot \vec{y} - (\vec{x} \cdot \vec{y}) \cdot \vec{z}$.

Exercise 9. Find the distance from the point $(3; 7; -5)$ to

(a) xy-plane

(b) yz-plane

(c) zx-plane

(d) x-axis

(e) y-axis

(f) z-axis

Exercise 10. Evaluate $\vec{a} + \vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$ and $|\vec{a} - \vec{b}|$.

(a) $\vec{a} = 4\vec{i} + \vec{j}$ and $\vec{b} = \vec{i} - 2\vec{j}$.

(b) $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = -2\vec{i} - \vec{j} + 5\vec{k}$.

Exercise 11. Find the angle between

(a) $\vec{a} = (3; 4)$ and $\vec{b} = (5; 12)$.

(b) $\vec{a} = (6; -3; 2)$ and $\vec{b} = (2; 1; -2)$.

Exercise 12. Find a unit vector that is orthogonal to both $\vec{i} + \vec{j}$ and $\vec{i} + \vec{k}$.

Exercise 13. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

Exercise 14. Prove that $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$.

Exercise 15. Find the volume of the parallelepiped determined by the vectors

(a) $\vec{a} = (6; -3; -1)$, $\vec{b} = (0; 1; 2)$ and $\vec{c} = (4; -2; 5)$.

(b) $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = -\vec{i} + \vec{j} + \vec{k}$.

Exercise 16. (a) Let P be a point not on the line that passes through the points Q, R and S. Show that the distance d from the point P to the plane (QRS) is

$$d = \frac{|\vec{a} \cdot (\vec{b} \times \vec{c})|}{|\vec{a} \times \vec{b}|},$$

where $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{QS}$ and $\vec{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2; 1; 4)$ to the plane through the points $Q(1; 0; 0)$, $R(0; 2; 0)$ and $S(0; 0; 3)$.

Exercise 17. Find an equation of the sphere with center $(1; -4; 3)$ and radius 5. Describe its intersection with each of the coordinate planes.

Exercise 18. Find an equation of the sphere that passes through the origin and whose center is $(1; 2; 3)$.

Exercise 19. Find an equation of a sphere if one of its diameters has end points $(2; 1; 4)$ and $(4; 3; 10)$.

Exercise 20. Find an equation of the largest sphere with center $(5; 4; 9)$ that is contained in the first octant.

Exercise 21. Consider the points P such that the distance from P to $A(-1; 5; 3)$ is twice the distance from P to $B(6; 2; -2)$. Show that the set of all such points is a sphere, and find its center and radius.

Exercise 22. Find an equation of the set of all points equidistant from the points $A(-1; 5; 3)$ and $B(6; 2; -2)$. Describe the set.

Exercise 23. Sketch and classify the quadric surface

(a) $x^2 + 2z^2 - 6x - y + 10 = 0$.

(b) $x = y^2 + 4z^2$.

(c) $x^2 = y^2 + 4z^2$.

(d) $-x^2 + 4y^2 - z^2 = 4$.

(e) $y = z^2 - x^2$.

(f) $4x^2 - 16y^2 + z^2 = 16$.

Exercise 24. Find an equation for the surface obtained by rotating

(a) the parabol $y = x^2$ about the y-axis.

(b) the line $x = 3y$ about the x-axis.

Exercise 25. Find an equation for the surface consisting of all points that are equidistant from the point $(-1; 0; 0)$ and the plane $x = 1$. Identify the surface.

Exercise 26. Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

Exercise 27. Change from rectangular to cylindrical coordinates.

(a) $(-1; 1; 1)$. (b) $(-2; 2\sqrt{3}; 3)$. (c) $(2\sqrt{3}; 2; -1)$. (d) $(4; -3; 2)$.

Exercise 28. Write the equations in cylindrical coordinates.

(a) $x^2 - x + y^2 + z^2 = 1$.

(b) $z = x^2 - y^2$.

(c) $3x + 2y + z = 6$.

(d) $-x^2 - y^2 + z^2 = 1$.

Exercise 29. Identify the surface whose equation is given

(a) $z = 4 - r^2$.

(b) $2r^2 + z^2 = 1$.

Exercise 30. Change from rectangular to spherical coordinates

(a) $(0; -2; 0)$. (b) $(-1; 1; -\sqrt{2})$. (c) $(1; 0; \sqrt{3})$. (d) $(\sqrt{3}; -1; 2\sqrt{3})$.

Exercise 31. Identify the surface whose equation is given.

(a) $r = \sin\theta \sin\varphi$.

(b) $r^2(\sin^2\varphi \sin^2\theta + \cos^2\varphi) = 9$.

Exercise 32. Write the equation in spherical coordinates

(a) $z^2 = x^2 + y^2$.

(b) $x^2 + z^2 = 9$.

(c) $x^2 - 2x + y^2 + z^2 = 0$.

(d) $x + 2y + 3z = 1$.

Chapter 2. Vector Functions

Exercise 33. Find the parametric equations for the intersection of the circular cylinder $x^2 + y^2 = 4$ and parabolic cylinder $z = x^3$.

Exercise 34. Find the domain.

(a) $\vec{r}(t) = (\sqrt{4 - t^2}, e^{-3t}, \ln(1 + t))$.

(b) $\vec{r}(t) = \dfrac{t-2}{t+2}\,\vec{i} + \sin t\,\vec{j} + \ln(9 - t^2)\,\vec{k}$.

(c) $\vec{r}(t) = \arcsin\dfrac{2t}{1+t}\,\vec{i} + \dfrac{\sqrt{t}}{\sin \pi t}\,\vec{k}$.

(d) $\vec{r}(t) = (\sqrt{\cosh t - 1}, \sqrt{t^4 - 5t^2 + 4}, 0)$.

Exercise 35. Find the limit

(a) $\lim\limits_{t\to 0}\left(\dfrac{e^t - 1}{t}, \dfrac{\sqrt{t+1} - 1}{t}, \dfrac{3}{t+1}\right)$.

(b) $\lim\limits_{t\to +\infty}\left(\arctan t, e^{-2t}, \dfrac{\ln t}{t+1}\right)$.

Exercise 36. Find a vector function that represents the curve of intersection of the two surfaces.

(a) The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

(b) The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

Exercise 37. Suppose u and v are vector functions that possess limits as $t \to a$ and let c be a constant. Prove the following properties of limits.

(a) $\lim\limits_{t\to a}[\vec{u}(t) + \vec{v}(t)] = \lim\limits_{t\to a}\vec{u}(t) + \lim\limits_{t\to a}\vec{v}(t)$.

(b) $\lim\limits_{t\to a} c\vec{v}(t) = c\lim\limits_{t\to a}\vec{v}(t)$.

(c) $\lim\limits_{t\to a}[\vec{u}(t) \cdot \vec{v}(t)] = \lim\limits_{t\to a}\vec{u}(t) \cdot \lim\limits_{t\to a}\vec{v}(t)$.

(d) $\lim\limits_{t\to a}[\vec{u}(t) \times \vec{v}(t)] = \lim\limits_{t\to a}\vec{u}(t) \times \lim\limits_{t\to a}\vec{v}(t)$.

Exercise 38. Find the derivative of the vector function.

(a) $\vec{r}(t) = (t\sin t, t^3, t\cos 2t)$.

(b) $\vec{r}(t) = \arcsin t\,\vec{i} + \sqrt{1-t^2}\,\vec{j} + \vec{k}$.

(c) $\vec{r}(t) = e^{t^2}\,\vec{i} - \sin^2 t\,\vec{j} + \ln(1+3t)\,\vec{k}$.

(d) $\vec{r}(t) = (e^{\sin t}, \arctan t, t^2)$.

Exercise 39. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

(a) $\begin{cases} x = t, \\ y = e^{-t}, \\ z = 2t - t^2, \end{cases}$ at $(0; 1; 0)$.

(b) $\begin{cases} x = t\cos t, \\ y = t, \\ z = t\sin t, \end{cases}$ at $(-\pi; \pi; 0)$.

Exercise 40. Find the point of intersection of the tangent lines to the curve $\vec{r}(t) = (\sin \pi t; 2\sin \pi t; \cos \pi t)$ at the points where $t = 0$ and $t = 0.5$.

Exercise 41. Evaluate the integral.

(a) $\int_0^{\pi/2}(3\sin^2 t\cos t\,\vec{i} + 3\sin t\cos^2 t\,\vec{j} + 2\sin t\cos t\,\vec{k})dt$.

(b) $\int_1^2(t^2\,\vec{i} + t\sqrt{t-1}\,\vec{j} + t\sin \pi t\,\vec{k})dt$.

(c) $\int_1^2(e^t\,\vec{i} + 2t\,\vec{j} + \ln t\,\vec{k})dt$.

(d) $\int_0^{1/4}(\cos \pi t\,\vec{i} + \sin \pi t\,\vec{j} + t^2\,\vec{k})dt$.

Exercise 42. If a curve has the property that the position vector $\vec{r}(t)$ is always perpendicular to the tangent vector $\frac{d}{dt}\vec{r}(t)$, show that the curve lies on a sphere with center the origin.

Exercise 43. Find the length of the curve.

(a) $\vec{r}(t) = (2\sin t, 5t, 2\cos t); -10 \le t \le 10$.

(b) $\vec{r}(t) = (2t, t^2, \frac{1}{3}t^3); 0 \le t \le 1$.

(c) $\vec{r}(t) = (\cos t, \sin t, \ln\cos t); 0 \le t \le \pi/4$.

(d) $\vec{r}(t) = (\sin t - t\cos t, \cos t + t\sin t, t^2);\ 0 \le t \le 2\pi$.

Exercise 44. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the length of C from the origin to the point $(6; 18; 36)$.

Exercise 45. Suppose you start at the point $(0; 0; 3)$ and move 5 units along the curve $x = 3\sin t; y = 4t; z = 3\cos t$ in the positive direction. Where are you now?

Exercise 46. Find the curvature of $\vec{r}(t) = (e^t \cos t, e^t \sin t, t)$ at the point $(1; 0; 0)$.

Exercise 47. Find the curvature of $\vec{r}(t) = (t, t^2, t^3)$ at the point $(1; 1; 1)$.

Exercise 48. Find the curvature of $y = \sqrt{x^2 + 1} - 2$ at the point $A(0; -1)$.

Exercise 49. Find the curvature of the curve given by $\begin{cases} x^2 + y^2 + 1 = 2(x - y), \\ x + y - z = 2 \end{cases}$ at the point $A(1; 0; -1)$.

Exercise 50. Find the curvature.

(a) $\vec{r}(t) = t\,\vec{i} + t\,\vec{j} + (1 + t^2)\,\vec{k}$.

(b) $\vec{r}(t) = 3t\,\vec{i} + 4\sin t\,\vec{j} + 4\cos t\,\vec{k}$.

(c) $x = e^t \cos t, y = e^t \sin t$.

(d) $x = t^3 + 1, y = t^2 + 1$.

Exercise 51. Find the curvature.

(a) $y = 2x - x^2$.

(b) $y = \cos x$.

(c) $y = 4x^{5/2}$.

(d) $y = \sin x$.

Exercise 52. At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$.

(a) $y = \ln x$.

(b) $y = e^x$.

Exercise 53. Find an equation of a parabola that has curvature 4 at the origin.

Exercise 54. Find the velocity vector, acceleration vector, and speed of a moving particle with the given position function

(a) $\vec{r}(t) = (e^{-t}, t\sqrt{3}, e^t)$.

(b) $\vec{r}(t) = e^t(\sin t, t, \cos t)$.

Exercise 55. A moving particle starts at an initial position $\vec{r}(0) = (2; -3; 4)$ with initial velocity $\vec{v}(0) = (1; 5; -4)$. Find its velocity and position at time t if $\vec{a}(t) = (2, t^3, e^{3t})$. Find its speed at $t = 1$.

Chapter 3. Double Integrals

Exercise 56. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) : -1 \leq x \leq 2, -1 \leq y \leq 1\}$.

Exercise 57. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0, y = \pi$ and $z = 0$.

Exercise 58. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

Exercise 59. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y-2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

Exercise 60. Evaluate the following integrals

(a) $\int_1^3 dx \int_1^5 \frac{\ln y}{xy} dx$.

(b) $\int_0^1 dx \int_0^1 xy\sqrt{x^2+y^2}dy$.

(c) $\iint_R \frac{1}{1+x+y} dxdy$, where $R = [1;3] \times [1;2]$.

(d) $\iint\limits_D ye^{xy}dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2; 0 \le y \le 3\}$.

(e) $\iint\limits_D |x+y|dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : |x| \le 1; |y| \le 1\}$.

(f) $\iint\limits_D \sqrt{|y - x^2|}dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : |x| \le 1, 0 \le y \le 2\}$.

(g) $\iint\limits_D |y - x^2|^3 dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : |x| \le 1, \text{ and } 0 \le y \le 2\}$.

Exercise 61. Use Midpoint rule to estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = \{(x,y) : 0 \le x \le 6, 0 \le y \le 4\}$. Use a Riemann sum with $m = 3, n = 2$.

Exercise 62.

(a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1;2] \times [0;3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.

(b) Use the Midpoint Rule to estimate the volume in part (1).

Exercise 63. A table of values is given for a function $f(x,y)$ defined on $R = [0;4] \times [2;4]$.

(a) Estimate $\iint_R f(x,y)dxdy$ using Midpoint rule with $m = n = 2$.

(b) Estimate the double integral with $m = n = 4$ by choosing the sample points to be the points closest to the origin.

x \ y	2.0	2.5	3.0	3.5	4.0
0	-3	-5	-6	-4	-1
1	-1	-2	-3	-1	1
2	1	0	-1	1	4
3	2	2	1	3	7
4	3	4	2	5	9

Exercise 64. Change the order of the following integration.

(a) $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)dy.$

(b) $\int\limits_{0}^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x,y)dx.$

(c) $\int\limits_{0}^{2} dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)dy.$

(d) $\int\limits_{0}^{\frac{\pi}{2}} dy \int\limits_{\sin y}^{1+y^2} f(x,y)dx.$

(e) $\int\limits_{0}^{\sqrt{2}} dy \int\limits_{0}^{y} f(x,y)dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_{0}^{\sqrt{4-y^2}} f(x,y)dx.$

Exercise 65. Evaluate the integrals

(a) $\int\limits_{0}^{1} dx \int\limits_{0}^{1-x^2} \dfrac{xe^{3y}}{1-y} dy.$

(b) $\iint\limits_{D} x^2(y-x)dxdy$, where D is bounded by $y = x^2$ and $x = y^2$.

(c) $\iint_D \dfrac{y}{1+x^2} dxdy$, where D is bounded by $y = \sqrt{x}, y = 0$ and $x = 1$.

(d) $\iint\limits_{D} xydxdy$, where D is bounded by $x = y^2, x = -1, y = 0$ and $y = 1$.

(e) $\iint\limits_{D} (x+y)dxdy$, where D is bounded by $x^2 + y^2 \leq 1, \sqrt{x} + \sqrt{y} \geq 1$.

(f) $\iint_D (x^2+y^2)^{3/2} dxdy$, where D is a region in the first quadrant and bounded by $y = 0$, $y = \sqrt{3}x$ and circle $x^2 + y^2 = 9$.

(g) $\iint\limits_{D} (|x| + |y|)dxdy$, $D = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$.

Exercise 66. Change to polar coordinates in a double integral $\iint\limits_{D} f(x,y)dxdy$, where D is a region as follows:

(a) $a^2 \leq x^2 + y^2 \leq b^2$.

(b) $x^2 + y^2 \geq 4x, x^2 + y^2 \leq 8x, y \geq x, y \leq \sqrt{3}x$.

(c) $x^2 + y^2 \leq 2x, x^2 + y^2 \leq 2y$.

Exercise 67. Use polar coordinates to find the following integrals

(a) $\int\limits_{0}^{R} dx \int\limits_{0}^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)dy, \quad (R > 0).$

(b) $\iint\limits_{D} xydxdy$, where D is a disk $(x-2)^2 + y^2 \leq 1, y \geq 0$.

(c) $\iint\limits_{D} (\sin y + 3x)dxdy$, where D is a disk $(x-2)^2 + y^2 \leq 1$.

(d) $\iint\limits_{D} |x+y|dxdy$, where D is a disk $x^2 + y^2 \leq 1$.

Exercise 68. Evaluate the following integrals:

(a) $\iint\limits_D \dfrac{2xy+1}{\sqrt{1+x^2+y^2}}dxdy$, with $D: x^2+y^2 \le 1$.

(b) $\iint\limits_D \dfrac{dxdy}{(x^2+y^2)^2}$, with $D: \begin{cases} y \le x^2+y^2 \le 2y \\ x \le y \le \sqrt{3}x. \end{cases}$

Exercise 69. Find the mass and center of mass of the lamina that occupies the region D and has the given density function $f(x,y)$.

(a) $D = \{(x,y) : 1 \le x \le 3, 1 \le y \le 4\}$, $f(x,y) = 2y^2$.

(b) $D = \{(x,y) : 0 \le x \le a, 0 \le y \le b\}$, $f(x,y) = 1 + x^2 + y^2$.

(c) D is bounded by $y = 1 - x^2$ and $y = 0$; $f(x,y) = ky$.

(d) D is bounded by $y = x^2$ and $y = x + 2$; $f(x,y) = kx$.

(e) $D = \{(x,y) : 0 \le y \le \sin\dfrac{\pi x}{L}, 0 \le x \le L\}$; $f(x,y) = y$.

(f) D is bounded by the parabolas $y = x^2$, and $x = y^2$; $f(x,y) = \sqrt{x}$.

Exercise 70. Find the area of the surface.

(a) The part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $[0;5] \times [1;4]$.

(b) The part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$.

(c) The part of the paraboloid $z = 4 - x^2 - y^2$ that lies above xy-plane.

(d) The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

Exercise 71. Use the change of variables $u = x + y$ and $v = x - y$ to evaluate the integral

$$\int_0^1 dx \int_{-x}^{x} (2 - x - y)^2 dy.$$

Exercise 72. Evaluate the following integrals:

(a) $\iint\limits_D \dfrac{xy}{x^2+y^2}dxdy$, where $D: \begin{cases} 2x \le x^2+y^2 \le 12 \\ x^2+y^2 \ge 2\sqrt{3}y \\ x \ge 0, y \ge 0. \end{cases}$

(b) $\iint\limits_D |9x^2 - 4y^2| dxdy$, where $D: \dfrac{x^2}{4} + \dfrac{y^2}{9} \le 1$.

Chapter 4. Triple Integrals

Exercise 73. Express the triple integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx$ in the order $dxdydz$.

Exercise 74. Evaluate the iterated integral

(a) $\int_0^1 \int_x^{2x} \int_0^y 2xyz \, dzdydx$,

(b) $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y dxdzdy$,

(c) $\int_0^1 \int_0^z \int_0^y ze^{-y^2} dxdydz$,

(d) $\int_0^{\frac{\pi}{2}} \int_0^y \int_0^x \cos(x+y+z) dzdxdy$.

Exercise 75. Evaluate the triple integral

(a) $\iiint\limits_E y dxdydz$, where E is bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

(b) $\iiint\limits_E x^2 e^y dxdydz$, where E is bounded by the parabolic cylinder $z = 1-y^2$ and the planes $z=0, x=1, x=-1$.

(c) $\iiint\limits_E xy dxdydz$, where E is bounded by the parabolic cylinder $y=x^2$, $x=y^2$ and the planes $z=0, z=x+y$.

(d) $\iiint\limits_E x dxdydz$, where E is the bounded by the paraboloid $x = 4y^2+4z^2$ and the plane $x=4$.

(e) $\iiint\limits_E (x^3+xy^2) dxdydz$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1-x^2-y^2$.

Exercise 76. Find the volume of the region E bounded by the paraboloids $z = x^2+y^2$ and $z = 36-3x^2-3y^2$.

Exercise 77. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

Exercise 78. Find the center of mass and the moments of inertia of the cubic $[1;2]\times[1;2]\times[1;2]$ if the density is $\rho(x,y,z) = x^2+y^2+z^2$.

Exercise 79. Find the center of mass and the moments of inertia of the tetrahedron with vertices $(0;0;0)$; $(1;0;0)$; $(0;1;0)$ and $(0;0;1)$ if the density is C (constant $\neq 0$).

Exercise 80. Evaluate the integrals by changing to cylindrical coordinates.

(a) $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz dzdxdy$,

(b) $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} dzdydx$.

Exercise 81. Evaluate the triple integrals.

(a) $\iiint\limits_B x \, dxdydz$, where B is bounded by the cone $x = \sqrt{y^2+z^2}$, and the plane $x=1$.

(b) $\iiint\limits_B \sqrt{x^2+4z^2} \, dxdydz$, where B is bounded by the cone $x^2+4z^2=y^2$, and the plane $y=-1$.

Exercise 82. Evaluate the triple integrals.

(a) $\iiint\limits_A \sqrt{x^2+4y^2+z^2}\,dxdydz$, where A is given by $x^2+4y^2+z^2 \le 2x$.

(b) $\iiint\limits_A x^2\,dxdydz$, where A is bounded by the xz-plane and the hemispheres $y = \sqrt{9-x^2-z^2}$ and $y = \sqrt{16-x^2-z^2}$.

Exercise 83. Find the moments of inertia of the ball $B = \{x^2+y^2+z^2 \le 1\}$ if the density $\rho(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Exercise 84. Evaluate the integral by changing to spherical coordinates $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \int\limits_{x^2+y^2}^{\sqrt{2-x^2-y^2}} xydzdydx$.

Exercise 85. Let E be the solid given by $x \le x^2+y^2+z^2 \le 2x$, $y \le x^2+y^2+z^2 \le 2y$, and $z \le x^2+y^2+z^2 \le 2z$.

(a) Evaluate the Jacobian of doing change the variables:

$$u = \frac{x}{x^2+y^2+z^2}, \quad v = \frac{y}{x^2+y^2+z^2}, \quad w = \frac{z}{x^2+y^2+z^2}.$$

(b) Evaluate the triple integral $\iiint\limits_E \frac{1}{(x^2+y^2+z^2)^3} dxdydz$.

Exercise 86. Let E be the solid given by $|x-y| + |x+3y| + |x+y+z| \le 1$. Evaluate the triple integral $\iiint\limits_E xydxdydz$.

Chapter 5. Line Integrals

Exercise 87. Evaluate the following line integrals:

(a) $\int\limits_C xyds$, where $C : x = t^2, y = 2t, 0 \le t \le 1$.

(b) $\int\limits_C xy^4 ds$, where $C : x^2+y^2 = 9$, $x \ge 0$.

(c) $\int\limits_C (x^2y^3 - \sqrt{x})dy$, where C is the arc of the curve $y = \sqrt{x}$ from $(1;1)$ to $(4;2)$.

(d) $\int\limits_C x^2dx + y^2dy$, where C consists of circle $x^2+y^2 = 4$ from $(2;0)$ to $(0;2)$ and the segment from $(0;2)$ to $(4;3)$.

(e) $\int\limits_C (3x-y)ds$, where C is the half of circle $y = \sqrt{9-x^2}$.

(f) $\int\limits_C (x-y)ds$, where C is a circle $x^2+y^2 = 2x$.

(g) $\int\limits_C y^2ds$, where C is given by $x = a(t-\sin t), y = a(1-\cos t)$ with $0 \le t \le 2\pi, a > 0$.

(h) $\int\limits_C \sqrt{x^2+y^2}ds$, where C is a curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, with $0 \le t \le 2\pi, a > 0$.

Exercise 88. Evaluate the following line integrals:

(a) $\int\limits_C (x^2+y^2+z^2)ds$, where $C : x = t, y = \cos 2t, z = \sin 2t, 0 \le t \le 2\pi$.

(b) $\int\limits_C xe^{yz}ds$, where C is the segment from $(0;0;0)$ to $(1;2;3)$.

(c) $\int_C ydx + zdy + xdz$, where $C : x = \sqrt{t}, y = t, z = t^2, 1 \le t \le 4$.

(d) $\int_C (y+z)dx + (x+z)dy + (x+y)dz$, where C consists of two line segments from $(0;0;0)$ to $(1;0;1)$, and from $(1;0;1)$ to $(0;1;2)$.

Exercise 89. Evaluate the following line integrals

(a) $\int\limits_{AB} (x^2-2xy)dx + (2xy-y^2)dy$, where AB is a part of parabol $y = x^2$ from $A(1;1)$ to $B(2;4)$.

(b) $\int\limits_C (2x-y)dx + xdy$, where C is a curve $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$ whose direction is increasing direction of the parameter t, $(0 \le t \le 2\pi, a > 0)$.

(c) $\int\limits_{ABCA} 2(x^2+y^2)dx + x(4y+3)dy$, where $ABCA$ is a broken line through the points $A(0;0)$, $B(1;1)$, $C(0;2)$.

(d) $\int\limits_{ABCDA} \dfrac{dx+dy}{|x|+|y|}$, where $ABCDA$ is a broken line through the points $A(1;0)$, $B(0;1)$, $C(-1;0)$, $D(0;-1)$.

(e) $\int\limits_C \dfrac{\sqrt[4]{x^2+y^2}dx}{2} + dy$, where C is curve $\begin{cases} x = t\sin\sqrt{t} \\ y = t\cos\sqrt{t}, (0 \le t \le \dfrac{\pi^2}{4}). \end{cases}$

Exercise 90. Evaluate the following line integral

$$\int\limits_C (xy+x+y)dx + (xy+x-y)dy$$

in two ways: by computing it directly, and by Green's formula, then compare the results, where C is a curve:

(a) $x^2+y^2 = 2x$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, (a, b > 0)$

Exercise 91. Evaluate the following line integrals:

(a) $\oint\limits_{x^2+y^2=2x} x^2\left(y+\dfrac{x}{4}\right)dy - y^2\left(x+\dfrac{y}{4}\right)dx.$

(b) $\oint\limits_{OABO} e^x[(1-\cos y)dx - (y-\sin y)dy]$, where $OABO$ is a broken line through the points $O(0;0)$, $A(1;1)$, $B(0;2)$.

(c) $\oint\limits_{x^2+y^2=2x} (xy + e^x \sin x + x + y)dx - (xy - e^{-y} + x - \sin y)dy.$

(d) $\oint\limits_C (xy^4 + x^2 + y\cos(xy))dx + \left(\dfrac{x^3}{3} + xy^2 - x + x\cos(xy)\right) dy$, where C is a curve $x = a\cos t, y = a\sin t$, $(a > 0)$.

(e) $\oint_C (e^x + y^6)dx + (e^y + 3x)dy$, where C is a boundary of region enclosed by $x = 14 + \sqrt{|y|}$ and $x = y^2$, with oriented counterclockwise.

Exercise 92. Using the line integral of the second kind in order to compute the area of the region bounded by an arch of the cycloid: $x = a(t - \sin t), y = a(1 - \cos t)$ and x-axis, $(a > 0)$.

Exercise 93. Evaluate the following line integral

(a) $\displaystyle\int\limits_{(-2;-1)}^{(3;0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy.$

(b) $\displaystyle\int\limits_{(1;\pi)}^{(2;2\pi)} (1 - \frac{y^2}{x^2}\cos\frac{y}{x})dx + (\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x})dy.$

Exercise 94. Evaluate the line integral

$$I = \int\limits_L (3x^2y^2 + \frac{2}{4x^2+1})dx + (3x^3y + \frac{2}{y^3+4})dy,$$

where L is curve $y = \sqrt{1 - x^4}$ from $A(1; 0)$ to $B(-1; 0)$.

Exercise 95. Find the constant α such that the following integral is an independent of path in the domain

$$\int\limits_{AB} \frac{(1 - y^2)dx + (1 - x^2)dy}{(1 + xy)^\alpha}.$$

Exercise 96. Find the curl and the divergence of the vector

(a) $\overrightarrow{F}(x, y, z) = xy\vec{i} + yz\vec{j} + zx\vec{k}.$

(b) $\overrightarrow{F}(x, y, z) = \dfrac{x}{x^2+y^2+z^2}\vec{i} + \dfrac{y}{x^2+y^2+z^2}\vec{j} + \dfrac{z}{x^2+y^2+z^2}\vec{k}.$

Exercise 97. Prove that

(a) $\text{curl}(\overrightarrow{F} + \vec{G}) = \text{curl}\overrightarrow{F} + \text{curl}\vec{G}.$

(b) $\text{curl}(f\overrightarrow{F}) = f\text{curl}\overrightarrow{F} + (\nabla f) \times \overrightarrow{F}.$

Exercise 98. Determine whether or not $\overrightarrow{F}$ is a conservative vector field. If it is, find a function f such that $\overrightarrow{F} = \nabla f$.

(a) $\overrightarrow{F}(x, y) = (2x - 3y)\vec{i} + (-3x + 4y - 8)\vec{j}.$

(b) $\overrightarrow{F}(x, y) = e^x \cos y\vec{i} + e^x \sin y\vec{j}.$

(c) $\overrightarrow{F}(x, y) = (xy\ cos(xy) + \sin(xy))\vec{i} + (x^2\cos(xy))\vec{j}.$

(d) $\overrightarrow{F}(x,y) = (\ln y + 2xy^3)\vec{i} + (3x^2y^2 + \dfrac{x}{y})\vec{j}$.

Exercise 99. Find f such that $\overrightarrow{F} = \nabla f$ and then compute $\int_C \overrightarrow{F} \cdot d\vec{r}$.

(a) $\overrightarrow{F}(x,y) = xy^2\vec{i} + x^2y\vec{j}$, where $C : \vec{r}(t) = (t + \sin\dfrac{\pi t}{2}, t + \cos\dfrac{\pi t}{2}), 0 \le t \le 1$.

(b) $\overrightarrow{F}(x,y) = \dfrac{y^2}{1+x^2}\vec{i} + 2y\arctan x\vec{j}$, where $C : \vec{r}(t) = t^2 i + 2tj, 0 \le t \le 1$.

(c) $\overrightarrow{F}(x,y) = (2xz + y^2)\vec{i} + 2xy\vec{j} + (x^2 + 3z^2)\vec{k}$, where $C : x = t^2, y = t+1, z = 2t-1, 0 \le t \le 1$.

(d) $\overrightarrow{F}(x,y) = e^y\vec{i} + xe^y\vec{j} + (z+1)e^z\vec{k}$, where $C : x = t, y = t^2, z = t^3, 0 \le t \le 1$.

Chapter 6. Surface Integrals

Exercise 100. Evaluate the surface integrals of scalar fields.

(a) $\iint\limits_F xy\,dS$, where F is the triangular region with vertices $(1;0;0)$, $(0;2;0)$, and $(0;0;2)$.

(b) $\iint\limits_F yz\,dS$, where F is the part of the plane $x+y+z=1$ that lies in the first octant.

(c) $\iint\limits_F yz\,dS$, where F is the surface with parametric equations $x = u^2$, $y = u\sin v$, $z = u\cos v$, $0 \le u \le 1$, $0 \le v \le \dfrac{\pi}{2}$.

(d) $\iint\limits_F z\,dS$, where F is the surface $x = y + 2z^2$, $0 \le y \le 1$, $0 \le z \le 1$.

(e) $\iint\limits_F y^2\,dS$, where F is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

(f) $\iint\limits_F \dfrac{dS}{(2+x+y+z)^2}$, where F is the boundary of the triangular pyramid $x+y+z \le 1; x \ge 0; y \ge 0; z \ge 0$.

Exercise 101. Find the area of

(a) the ellipse cut from the plane $z = 2x + y$ by the cylinder $x^2 + y^2 = 1$.

(b) the surface $x^2 - 2\ln x + \sqrt{15}y - z = 0$ above the square $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2, 0 \le y \le 1\}$ in the xy-plane.

(c) the part of the paraboloid $z = x^2 + y^2$ which lies under the plane $z = 6$.

(d) the surface determined by the parametric equations $x = z(\cos u + u\sin u)$, $y = z(\sin u - u\cos u)$, $0 \le u, z \le 1$.

Exercise 102. Find the mass of the surface F determined by the parametric equations $x = uv, y = u+v, z = u-v, u^2+v^2 \le 1, u \ge 0, v \ge 0$ if the density $\rho(x,y,z) = x + yz$.

Exercise 103. Find the center of mass of

(a) a thin hemisphere of radius R and constant mass density C.

(b) the triangle with vertices $(1;0;0)$, $(0;1;0)$, $(0;0;1)$ and the density $\rho(x,y,z) = x + 2y + z$.

(c) the cylinder $x^2+y^2 = 1$, $0 \le z \le 1$ and the density $\rho(x,y,z) = x^2+y^2+z^2$.

Exercise 104. Evaluate the surface integral $\iint\limits_A \overrightarrow{F} \cdot \vec{n}\, dS$ for the given vector field $\overrightarrow{F}$ and the oriented surface A. For closed surfaces, use the positive (outward) orientation.

(a) $\overrightarrow{F}(x,y,z) = xze^y\,\vec{i} - xze^y\,\vec{j} + z\,\vec{k}$, A is the part of the plane $x+y+z=1$ in the first octant and has downward orientation.

(b) $\overrightarrow{F}(x,y,z) = x\,\vec{i} + y\,\vec{j} + z^4\,\vec{k}$, A is the part of the cone $z = \sqrt{x^2+y^2}$ beneath the plane $z = 1$ with downward orientation.

(c) $\overrightarrow{F}(x,y,z) = xz\,\vec{i} + x\,\vec{j} + y\,\vec{k}$, A is the hemisphere $x^2+y^2+z^2 = 25$; $y \ge 0$, oriented in the direction of the positive y-axis.

(d) $\overrightarrow{F}(x,y,z) = xy\,\vec{i} + 4x^2\,\vec{j} + yz\,\vec{k}$, A is the surface $z = xe^y$, $0 \le x, y \le 1$, with upward orientation.

(e) $\overrightarrow{F}(x,y,z) = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$, A is the boundary of the solid half-cylinder $0 \le z \le \sqrt{1-y^2}$, $0 \le x \le 2$.

(f) $\overrightarrow{F}(x,y,z) = (x,y,z)$ and A is the upper surface, upward oriented, $z = 16 - x^2 - y^2$, $x^2+y^2 \le 16$.

(g) $\overrightarrow{F}(x,y,z) = (\frac{x}{z}, \frac{y}{z}, z-2)$ and A is the upper surface, upward oriented, of $z = 4 - x^2 - y^2$, $x^2+y^2 \le 2$.

(h) $\overrightarrow{F}(x,y,z) = (0, y, -z)$ and A consists of the paraboloid $y = x^2+z^2$, $0 \le y \le 1$ and the disk $x^2+z^2 \le 1$, $y = 1$.

Exercise 105. Evaluate the surface integral $\iint\limits_A \overrightarrow{F} \cdot \vec{n}\, dS$ for the given vector field $\overrightarrow{F}$ and the oriented surface A.

(a) $\overrightarrow{F}(x,y,z) = (x,z,y)$ and A is the sphere $x^2+y^2+z^2 = 1$, oriented outward.

(b) $\overrightarrow{F}(x,y,z) = (x,2y,3z)$ and A is the cube $[1;2] \times [1;2] \times [1;2]$, oriented outward.

(c) $\overrightarrow{F}(x,y,z) = (x+2y, 2y+3z, 3z+x)$ and A the triangular pyramid $ODBC$, $O(0;0;0)$, $D(1;0;0)$, $B(0;1;0)$, $C(0;0;1)$, oriented outward.

Exercise 106. A fluid with density C flows with velocity $\vec{v} = (yx^2, x, z)$. Find the rate of flow upward through the paraboloid $A : z = 9 - \dfrac{x^2+y^2}{4}$, $x^2+y^2 \le 36$.

Exercise 107. Let $\overrightarrow{F}$ be an inverse square field, that is $\overrightarrow{F}(\vec{r}) = C\dfrac{\vec{r}}{|\vec{r}|^3}$, for some constant C, where $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$. Show that the surface integral $\iint\limits_A \overrightarrow{F} \cdot \vec{n}\, dS$, where A is a sphere with center at the origin, is independent of the radius of A.

Exercise 108. Use Stokes' Theorem to evaluate $\int\limits_C \overrightarrow{F} \cdot d\vec{r}$. In each case C is oriented counterclockwise as viewed from above.

(a) $\overrightarrow{F}(x,y,z) = yz\,\vec{i} + 2xz\,\vec{j} + 3^{xy}\,\vec{k}$ and C is the circle $x^2+y^2=4$, $z=10$.

(b) $\overrightarrow{F}(x,y,z) = (3x+2y^2)\,\vec{i} + (8y+\dfrac{z^2}{3})\,\vec{j} + (4z+\dfrac{3x^2}{2})\,\vec{k}$ and C is the boundary of the triangle with vertices $(2;0;0)$, $(0;3;0)$ and $(0;0;6)$.

(c) $\overrightarrow{F}(x,y,z) = xy\,\vec{i} + 2z\,\vec{j} + 3y\,\vec{k}$ and C is the curve of intersection of the plane $x+z=5$ and the cylinder $x^2+y^2=9$.

(d) $\overrightarrow{F}(x,y,z) = x^2z\,\vec{i} + xy^2\,\vec{j} + z^2\,\vec{k}$ and C is the curve of intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$.

Exercise 109. The work done by the force field $\overrightarrow{F}(x,y,z) = (x^x+z^2, y^y+x^2, z^z+y^2)$ when a particle moves under its influence around the close edge of the part of the sphere $x^2+y^2+z^2=4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

Exercise 110. Use Stokes' Theorem to evaluate $\iint\limits_A \operatorname{curl}\overrightarrow{F} \cdot \vec{n}\,dS$.

(a) $\overrightarrow{F}(x,y,z) = 2y\cos z\,\vec{i} + e^x\sin z\,\vec{j} + xe^y\,\vec{k}$ and A is the hemisphere $x^2+y^2+z^2=9$, $z\geq 0$ oriented upward.

(b) $\overrightarrow{F}(x,y,z) = x^2z^2\,\vec{i} + y^2z^2\,\vec{j} + xyz\,\vec{k}$ and A is the part of the paraboloid $z=x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$, oriented upward.

(c) $\overrightarrow{F}(x,y,z) = (xyz, xy, x^2yz)$ and A consists of the top and the four sides but not the bottom of the cube $[0;1]\times[0;1]\times[0;1]$, oriented outward.

(d) $\overrightarrow{F}(x,y,z) = (e^xyz, y^2z, 2z)$, A is the part of the hemisphere $x^2+y^2+z^2=9$, $x\geq 0$, that lies inside the cylinder $y^2+z^2=4$, oriented in the direction of the positive x-axis.

Exercise 111. Use the Divergence Theorem to calculate $\iint\limits_A \overrightarrow{F} \cdot \vec{n}\,dS$.

(a) $\overrightarrow{F}(x,y,z) = x^3y\,\vec{i} - x^2y^2\,\vec{j} - x^2yz\,\vec{k}$ and A is the surface of the solid bounded by the hyperboloid $x^2+y^2-z^2=1$ and the planes $z=-2$ and $z=2$.

(b) $\overrightarrow{F}(x,y,z) = (\cos z + xy^2)\,\vec{i} + xe^{-z}\,\vec{j} + (\sin y + x^2z)\,\vec{k}$ and A is the surface of the solid bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$.

(c) $\overrightarrow{F}(x,y,z) = 4x^3z\,\vec{i} + 4y^3z\,\vec{j} + 3z^4\,\vec{k}$ and A is the sphere with radius R and center the origin.

(d) $\overrightarrow{F}(x,y,z) = z^2x\,\vec{i} + (y^3+\sin z)\,\vec{j} + (x^2z+y^2)\,\vec{k}$ and A is the upward oriented top half of the sphere $x^2+y^2+z^2=1$.

(e) $\overrightarrow{F}(x,y,z) = z^2y^{10}\,\vec{i} + (4x^2y^3+\sin z)\,\vec{j} + (2x^2z+y^2)\,\vec{k}$ and A is the outward oriented surface of the cube $[-1;1]\times[-1;1]\times[-1;1]$.

(f) $\overrightarrow{F}(x, y, z) = -xy\,\vec{i} + 3y^2\,\vec{j} + 3zy\,\vec{k}$ and A is the outward oriented surface of the tetrahedron with vertices $(0; 0; 0)$, $(1; 0; 0)$, $(0; 1; 0)$, and $(0; 0; 1)$.

(g) $\overrightarrow{F}(x, y, z) = 6xy^2\,\vec{i} + 3x^2e^{2z}\,\vec{j} + 2z^3\,\vec{k}$ and A is the outward oriented surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = -1$, and $z = 2$.

(h) $\overrightarrow{F}(x, y, z) = x^5\,\vec{i} + \frac{10}{3}x^2y^3\,\vec{j} + 5zy^4\,\vec{k}$ and A is the outward oriented surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.