Hanoi University of Science and Technology

School of Applied Mathematics and Informatics

CALCULUS I EXERCISE

COURSE ID: MI 1016

1.1-1.3. Functions. Essential functions

Exercise 1. Determine the domain of the following functions.

a) $y = \dfrac{x}{\sqrt{4x^2-1}}$.

b) $y = \arcsin \dfrac{x}{x+2}$.

c) $y = \ln \dfrac{1-x}{1+x}$.

d) $y = \sqrt{\arctan x}$.

Exercise 2. Find the range of the following functions.

a) $y = \ln(1-2\sin x)$.

b) $y = \arctan(2e^x)$.

c) $y = \sqrt{\arccos x}$.

d) $y = \dfrac{x^2-1}{x^2+1}$.

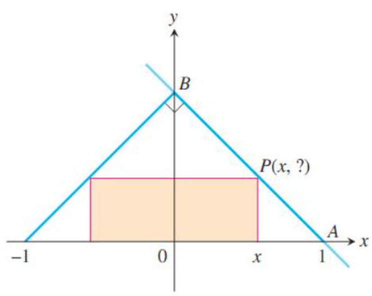
Exercise 3. As dry air moves upward, it expands and cools. The ground temperature is 30°C and the temperature at a height of 1 km is 20°C.

a) Express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

b) Draw the graph of the function.

c) What is the temperature at a height of 4 km?

Exercise 4. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



a) Express the y-coordinate of P in terms of x.

b) Express the area of the rectangle in terms of x.

Exercise 5. Determine whether f is even, odd, or neither.

a) $f(x) = \dfrac{e^x - e^{-x}}{2} =: \sinh x$.

b) $f(x) = \dfrac{2^x - x^2}{2^x + x^2}$.

c) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

d) $f(x) = \ln \dfrac{1-x}{1+x}$.

e) $f(x) = \sin x + \cos 2x$.

f) $f(x) = \sin x + \sin 2x$.

Exercise 6. Find the functions $f \circ g, g \circ f, f \circ f$, and $g \circ g$ and their domains.

a) $f(x) = \dfrac{1}{x+1}$, $g(x) = x - 1$.

b) $f(x) = \sqrt{2x+3}$, $g(x) = x^2 + 1$.

c) $f(x) = \sin x$, $g(x) = \sqrt{x+1}$.

d) $f(x) = 1 - x^2$, $g(x) = \dfrac{1-x}{1+x}$.

Exercise 7. Find the inverse functions of the following functions.

a) $y = \arcsin 2x$.

b) $y = \dfrac{e^{2x} - e^{-2x}}{2}$.

c) $y = \dfrac{1-3x}{1+3x}$.

d) $y = \ln \dfrac{e^x - 1}{e^x + 1}$.

2.1 - 2.3. Limits

Exercise 8. Find the limit of the following sequences (if it exists).

a) $u_n = \sqrt[3]{n^3 + 2n^2} - n$.

b) $u_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$.

c) $u_n = \cos\left(\dfrac{n\pi}{2}\right)$.

d) $u_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}$.

e) $u_n = \left(1 - \dfrac{1}{2n}\right)^n$.

f) $u_n = \dfrac{n\cos(n^2+1)}{n^2+2}$.

Exercise 9. Find the limit of the sequence $\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots\}$.

Exercise 10. Determine the function α which is infinite as $x \to \infty$ and the integer n (if there exists) such that α and x^n are of the same order.

a) $\alpha(x) = x^3 + 2x^2 + x^5$.

b) $\alpha(x) = 3x^4 + \sin x$.

c) $\alpha(x) = \sin(x^2)$.

d) $\alpha(x) = e^x + x^2$.

Exercise 11. Determine the function α which is infinitesimal as $x \to 0$ and the integer n (if there exists) such that α and x^n are of the same order.

a) $\alpha(x) = x^3 + 2x^2 + x^5$.

b) $\alpha(x) = 3x^4 + \sin x$.

c) $\alpha(x) = \sin(x^2)$.

d) $\alpha(x) = e^x + x^2$.

Exercise 12. Compare the order of the following infinitesimals as $x \to 0$.

a) $\alpha(x) = \sin(x^2 + x), \beta(x) = 1 - \cos x$.

b) $\alpha(x) = \sqrt{x^3 + 2x^4}, \beta(x) = \ln(1+2x)$.

c) $\alpha(x) = e^{\sin x} - 1, \beta(x) = \arcsin(\tan x)$.

d) $\alpha(x) = \tan(-x^2 + 3x^3), \beta(x) = \sinh(3x^2)$.

Exercise 13. Evaluate the following limits.

a) $\lim\limits_{x\to 1} \dfrac{\sqrt{x^2+8} - 3}{x-1}$.

b) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$.

c) $\lim\limits_{x\to 0} \dfrac{9^x - 5^x}{4^x - 3^x}$.

d) $\lim\limits_{x\to 1} \dfrac{x^6 - 1}{x^{10} - 1}$.

e) $\lim\limits_{x\to 0} \dfrac{\ln(1+2x^2)}{1-\cos x}$.

f) $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x+3}\right)^{2x}$.

g) $\lim\limits_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$.

h) $\lim\limits_{x\to 0}\frac{\ln(1+3\tan x)}{e^x-\cos x}$.

i) $\lim\limits_{x\to 0^+}\frac{\sqrt{x^5}}{\sqrt{\sin x}\ln(1-3x^2)}$.

j) $\lim\limits_{x\to 0}\frac{\cos(\sin x)-1}{\sin(\cos x-1)}$.

Exercise 14. If $\lim\limits_{x\to 1}\frac{f(x)-8}{x-1}=9$, find $\lim\limits_{x\to 1}f(x)$.

3.1 - 3.6. Continuity of functions

Exercise 15. For what value of a is $f(x)=\begin{cases} x^2+2, \text{ if } x<1 \\ 2ax^3+1, \text{ if } x\geq 1\end{cases}$ continuous at every x?

Exercise 16. Show that f is continuous on $(-\infty,\infty)$.

a) $f(x)=\begin{cases}\sin x & \text{if } x<\frac{\pi}{4} \\ \cos x & \text{if } x\geq\frac{\pi}{4}\end{cases}$.

b) $f(x)=\begin{cases}x^2 & \text{if } x<1 \\ \sqrt{x} & \text{if } x\geq 1\end{cases}$.

Exercise 17. Locate the discontinuity of the function and illustrate by graphing

a) $y=\frac{1}{1+e^{1/x}}$.

b) $y=\ln(\tan^2 x)$.

c) $y=\frac{\sin x}{2^x-1}$.

Exercise 18. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither?

a) $f(x)=\begin{cases}\frac{2^x-1}{x} & \text{if } x<0 \\ 2x+c & \text{if } x\geq 0\end{cases}$.

b) $f(x)=\begin{cases}\frac{\sin^2(\pi x)}{\ln(1+2x^2)} & \text{if } x<1 \\ \sqrt{x} & \text{if } x\geq 1\end{cases}$.

Exercise 19. Prove that there is a root of the given equation in the specified interval.

a) $x^6-3x+1=0, \quad (0,1)$.

b) $x^3=\sqrt{3x+1}, \quad (1,2)$.

Exercise 20. A train starts at 8AM from Hanoi to Haiphong, arriving at 11AM. The next day it starts at 8AM from Haiphong to Hanoi, arriving at 11AM. Is there a point on the route the train will cross at exactly the same time of day on both days?

Exercise 21. Let f be a continuous function on a close interval $[0,1]$ and $f(0)=1, f(1)=0$. Prove that there is a number $c\in(0,1)$ at which $f(c)=c$.

4.1 - 4.6. Derivatives

Exercise 22. Find the derivative of the following functions:

a) $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$.

b) $y = \sin(\tan x)$.

c) $y = \sqrt{x + \sqrt{x}}$.

d) $y = \ln\left(x + \sqrt{x^2 + 5}\right)$.

e) $y = \sin^n x \cos nx$.

f) $y = \left(1 + \dfrac{1}{x}\right)^x$.

Exercise 23. For what values of a and b will

$$f(x) = \begin{cases} ax & \text{if } x < 2 \\ ax^2 + bx + 3 & \text{if } x \geq 2 \end{cases}$$

be differentiable for all values of x? Discuss the geometry of the resulting graph of f.

Exercise 24. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$. Find the values of m and b that make f differentiable everywhere.

Exercise 25. Let $r(x) = f(g(h(x)))$, where $h(1) = 2, g(2) = 3, h'(1) = 4, g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

Exercise 26. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.

Exercise 27. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answer.

Exercise 28. Find $y'(x)$ if y is defined implicitly as a function of x by the equation

a) $\arctan(2x + y) = y^3$.

b) $x^3 + y^3 = 3x^2 y$.

c) $\cos(x - y) = xe^y$.

Exercise 29. Find the equation of the tangent line of the curve $2x^3 + 4y^2 = 6$ at the point $(1, 1)$.

Exercise 30. Find $x'(y)$ if x is defined implicitly as a function of y by the equation $y^2 + 2x^3y + x = 0$. Apply the previous computation to find the equation of the tangent line of the curve $y^2 + 2x^3y + x = 0$ at the point $(1, -1)$.

Exercise 31. Find the n - th derivatives of the following functions:

a) $y = \dfrac{1}{x^2 + x}$.

b) $y = \dfrac{x}{x^2 - 4}$.

c) $y = (x^2 + 1)\, e^{2x}$.

d) $y = \ln(2x^2 + x)$.

e) $y = (2x + 1)\cos 3x$.

f) $y = \cos(2x)\sin x$.

Exercise 32. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

Exercise 33. Prove each of the following.

a) The derivative of an even function is an odd function.

b) The derivative of an odd function is an even function.

Exercise 34. Find the derivative of the function $f(x) = \begin{cases} x \arctan \dfrac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$.

Exercise 35. Suppose that the functions f and g are defined throughout an open interval containing the points x_0, that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0.

5.1 - 5.5. Applications of Derivatives and Differentials

Exercise 36. Use a linear approximation (or differentials) to estimate the given number.

a) $\sqrt{1.002}$.

b) $\sqrt[3]{8.001}$.

c) $\sin(0.01)$.

d) $(1.999)^4$.

Exercise 37. Find equations of the tangent line and the normal line to the curve

a) $y = \ln\left(x + \sqrt{x^2 + 3}\right)$ at $x = 1$.

b) $y = x + \tanh(2x)$ at $x = 0$.

Exercise 38. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

Exercise 39. Find all the point on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent of the curve at such a point is

a) perpendicular to the line $y = 1 - \dfrac{x}{24}$,

b) parallel to the line $y = \sqrt{2} - 12x$.

Exercise 40. Show that the tangents to the curve $y = \dfrac{\pi \sin x}{x}$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

Exercise 41. Given $f(x) = \ln \dfrac{x+1}{x+2}$, find $df(x), d^{10}f(x)$.

Exercise 42. Given $f(x) = (x + 2)\ln x$, find $d^2 f(1), d^{20} f(1)$.

Exercise 43. Find the n th-degree Taylor polynomials centered at $x = 0$ of $f(x)$. Determine the remainder.

a) $f(x) = x \cos x, n = 5$.

b) $f(x) = \dfrac{x}{\sqrt{1 + x^2}}, n = 5$.

c) $f(x) = \sqrt{2 + 2x}, n = 3$.

d) $f(x) = e^{2x} + 1, n = 4$.

5.6 - 5.9. Applications of Derivatives and Differentials

Exercise 44. Evaluate the following limits

a) $\lim\limits_{x\to 0} \dfrac{\arcsin x - x}{x^3}$.

b) $\lim\limits_{x\to 0} \dfrac{x^5 - \ln(1 + x^5)}{\sin^{10} x}$.

c) $\lim\limits_{x\to 0} x \ln |x|$.

d) $\lim\limits_{x\to +\infty} x[\pi - 2\arctan(3x)]$.

e) $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$.

f) $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{2}{e^{2x} - 1}\right)$.

g) $\lim\limits_{x\to -\infty} (x^2 + 2^x)^{\frac{1}{x}}$.

h) $\lim\limits_{x\to 0} [\ln(e + 2x)]^{\frac{1}{\sin x}}$.

i) $\lim\limits_{x\to 0} [2x + e^{3x}]^{\frac{1}{\sin x}}$.

j) $\lim\limits_{x\to 0^+} [\arcsin 2x]^{\tan x}$.

k) $\lim\limits_{x\to 0} \dfrac{\sin x \ln(x + 1) - x^2}{x^3}$.

l) $\lim\limits_{x\to 0^+} x^{\sin x}$.

Exercise 45. Show that

a) $\sin(\arccos x) = \cos(\arcsin x) = \sqrt{1-x^2}$ for all $x \in [-1,1]$.

b) $\dfrac{1}{2}\arctan\dfrac{2x}{1-x^2} = \arctan x$ for all $x \in (-1,1)$.

c) $\arcsin(\tanh x) = \arctan(\sinh x)$.

Exercise 46. Prove that

a) $|\arcsin x - \arcsin y| \geq |x-y|$ for all $x, y \in [-1,1]$.

b) $\dfrac{y-x}{1+y^2} < \arctan y - \arctan x < \dfrac{y-x}{1+x^2}$ for all $0 < x < y$.

c) $\dfrac{1}{2} - \dfrac{x}{8} < \dfrac{1}{x} - \dfrac{1}{e^x - 1} < \dfrac{1}{2}$ for all $x > 0$.

d) $\dfrac{x}{x+1} \leq \ln(x+1) \leq x$ for all $x > -1$.

Exercise 47. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

Exercise 48. Show that the equation $a\cos x + b\cos 2x + c\cos 3x = 0$ has at least one root on $(0, \pi)$.

Exercise 49. Suppose that $f(x)$ is a continuous function on a close interval $[a,b]$ and differentiable on (a,b), and $f(a) = f(b) = 0$. Show that there exists a number $c \in (a,b)$ such that $f'(c) = 2021f(c)$.

Exercise 50. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0.$$

Exercise 51. Determine the local extreme values

a) $y = x^{2/3}(x+2)$.

b) $y = x^{2/3}\,(x^2-4)$.

c) $y = x\sqrt{4-x^2}$.

d) $y = x^2\ln x$.

e) $y = (x^2-3)\,e^x$.

f) $y = 3\arctan x - \ln(x^2+1)$.

g) $y = \ln(x+3) + \operatorname{arccot} x$.

Exercise 52. Find the absolute maximum and minimum values of $f(x) = x^2 + \dfrac{250}{x}$ over $[1, 10]$.

Exercise 53. Find the intervals of concavity and the inflection points of the following functions.

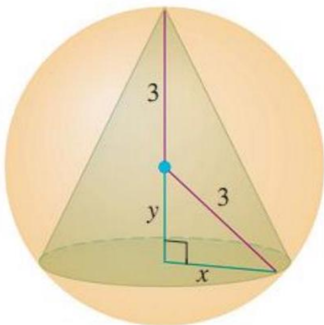
a) $f(x) = x^2 \ln x$.

b) $f(x) = (x+1)e^{-x}$.

c) $f(x) = (1-x)\sqrt[3]{x}$.

Exercise 54. (The best fencing plan) A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

Exercise 55. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



Exercise 56. (Designing a can) What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3.

Exercise 57. A rectangle is to be inscribed under the arch of the curve $y = 4\cos(x/2)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

Exercise 58. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

Exercise 59. Determine the asymptotes of the graph of $y = f(x)$.

a) $y = \dfrac{e^x}{x+1}$.

b) $y = x \operatorname{arccot} \dfrac{2}{x}$.

c) $y = (x+2)e^{1/x}$.

d) $y = \sqrt[3]{x^3 + x}$.

e) $y = e^x \ln x$.

Exercise 60. Determine the asymptotes of the curves.

a) $x = t^3 - 3\pi, y = t^3 - 6\arctan t.$

b) $x = \dfrac{t^2}{t-1}, y = \dfrac{t}{t^2-1}.$

Exercise 61. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x\to 0} \frac{f(2+3x) - f(2+5x)}{x}.$$

6.1 - 6.5. Indefinite Integrals 1

Exercise 62. Find the function f.

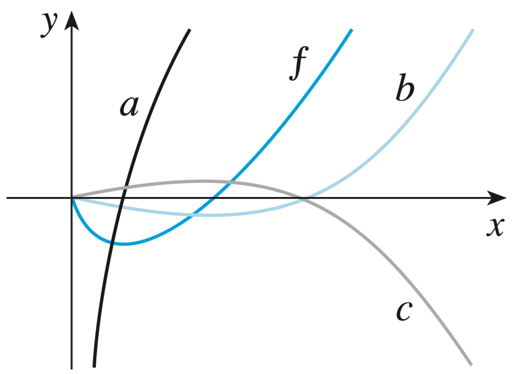
a $f'(x) = 1 + x, f(0) = 1.$

b $f'(x) = 5x^4 - 3x^2 + 4, f(-1) = 2.$

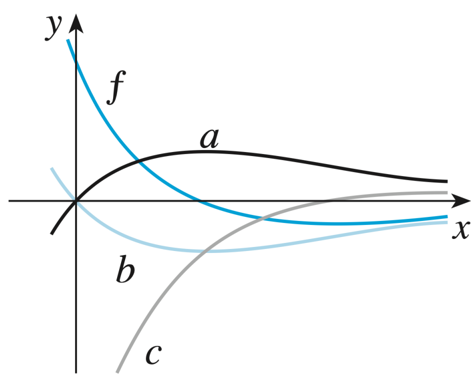
c $f''(x) = -2 - 12x^2, f(0) = 4, f'(0) = 12.$

d $f''(x) = 20x^3 + 12x^2, f(0) = 0, f'(0) = 1.$

Exercise 63. The graph of a function f is shown. Which one (a,b or c) is the anti-derivative of f? Give your explanation.



a)



b)

Exercise 64. What constant acceleration is required to increase the speed of a car from 30 km/h to 50 km/h in 5s?

Exercise 65. Find a function $f(x)$ such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of $f(x)$.

Exercise 66. Evaluate the following integrals

a) $\int x\sin(x^2)dx$.

b) $\int \frac{x+1}{x^2+2x+2}dx$.

c) $\int \frac{1}{x\ln^2 x}dx$.

d) $\int \frac{x}{\sqrt{x+1}}dx$.

e) $\int x\sin x dx$.

f) $\int x^2e^x dx$.

g) $\int \tan 2x dx$.

h) $\int e^x \sin x dx$.

6.6 - 6.8. Indefinite Integrals 2

Exercise 67. Evaluate the following integrals

a) $\int \frac{x^3+1}{x^2+4}dx$.

b) $\int \tan^4 x dx$.

c) $\int \frac{1-2x}{\sqrt{2+x^2}}dx$.

d) $\int \frac{x}{(x^2+1)(x+2)}dx$.

e) $\int \frac{\sin 2x}{\sqrt{\sin^4 x+1}}dx$.

f) $\int \frac{dx}{3\sin x-4\cos x}$.

g) $\int \frac{dx}{1+\sqrt{x^2+4x+5}}$.

h) $\int \frac{x+1}{\sqrt{x^2-2x-1}}dx$.

Exercise 68. Evaluate the following integrals

a) $\int (x+1)\arctan x dx$.

b) $\int (x+2)\ln x dx$.

c) $\int \arcsin^2 x dx$.

d) $\int \frac{\arctan x}{x^2}dx$.

e) $\int \frac{x}{(x^2+2x+2)^2}dx$.

f) $\int \frac{e^{2x}}{1+e^x}dx$.

g) $\int \sqrt{\frac{x}{x-1}}dx$.

h) $\int \frac{x^2+2}{x^3-1}dx$.

i) $\int \frac{x^2+1}{x^4+1}dx$.

j) $\int \frac{\sin^2 x}{\cos^3 x}dx$.

k) $\int \frac{1}{x^2\sqrt{x^2+1}}dx$.

7.1 - 7.5. Definite Integrals 1

Exercise 69. Show that

a) $\dfrac{13}{42} < \int_0^1 \sin(x^2)\,dx < \dfrac{1}{3}$.

b) $\int_0^{\pi/6} \cos(x^2)\,dx > \dfrac{1}{2}$.

Exercise 70. Find the derivative of the following functions.

a) $f(x) = \int_0^x \sqrt{1+t^4}dt$.

b) $g(x) = \int_0^{x^2} \sin(t^2)\,dt$.

c) $h(x) = \int_{x^3}^0 \sin^3 t dt$.

d) $k(x) = \int\limits_{x^2}^{x^3} \arcsin(t)dt$.

Exercise 71. Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on $[0,1]$.

a) $\lim\limits_{n\to\infty} \sum_{k=1}^n \dfrac{k^4}{n^5}$.

b) $\lim\limits_{n\to\infty} \dfrac{1}{n}\left(\sqrt{1+\dfrac{1}{n}} + \sqrt{1+\dfrac{2}{n}} + \cdots + \sqrt{1+\dfrac{n}{n}}\right)$.

Exercise 72. Evaluate the following integrals.

a) $\int_1^e (x\ln x)^2 dx$.

b) $\int_0^{\pi/4} \dfrac{\sin^2 x\cos x}{(1+\tan^2 x)^2}dx$.

c) $\int_1^2 \dfrac{\sqrt{x^2-1}}{x^2}dx$.

d) $\int_0^1 \dfrac{\ln(x^2+1)}{(x+1)^2}dx$.

Exercise 73. If f is continuous on $[0,1]$, show that

a) $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$.

b) $\int_0^{\pi} xf(\sin x)dx = \dfrac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

Exercise 74. Evaluate

a) $\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx$.

b) $\int_0^{\pi/2} \dfrac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx$.

7.6 - 7.8. Definite Integrals 2

Exercise 75. Find the area of the region enclosed by the parabolas $x = 2y - y^2, x = y^2 - 4y$.

Exercise 76. Find the area of the region enclosed by the curve $y^2 = x^2 - x^4$.

Exercise 77. Find the area of the region enclosed by $y = \dfrac{1}{x}, y = x$ and $y = \dfrac{1}{4}x, x > 0$.

Exercise 78. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Exercise 79. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

a) $y = 2x - x^2, y = 0$; about the x-axis.

b) $y = \ln x, y = 1, y = 2, x = 0$; about the y-axis.

c) $x = y^2, x = 1$; about $x = 1$.

d) $y = x^2, x = y^2$; about $y = -1$.

Exercise 80. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about

a) the x-axis.

b) the y-axis.

c) the line $x = 5$.

Exercise 81. Find the length of the curves

a) $y = \dfrac{x^2}{8} - \ln x, 4 \le x \le 8$.

b) $x = y^{2/3}, 1 \le y \le 8$.

c) $x = 5\cos t - \cos 5t, y = 5\sin t - \sin 5t, 0 \le t \le \pi/2$.

Exercise 82. Find the area of the surface generated by revolving the curve

a) $y = \sqrt{x^2 + 2}, 0 \le x \le \sqrt{2}$, about the x-axis.

b) $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x, 1 \le x \le 2$, about the y-axis.

7.9 - 7.10. Definite Integrals 3

Exercise 83. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a) $\int_0^\infty \frac{x}{(x^2+2)^2} dx.$

b) $\int_1^\infty \frac{x+2}{x^2+3x} dx.$

c) $\int_{-\infty}^\infty \frac{x}{x^2+1} dx.$

d) $\int_{-\infty}^0 xe^{-x} dx.$

e) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx.$

f) $\int_0^\infty \frac{e^x}{e^{2x}+3} dx.$

g) $\int_0^\infty \frac{x \arctan x}{(1+x^2)^2} dx.$

h) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}} dx.$

i) $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx.$

Exercise 84. Determine whether the improper integral is convergent or divergent.

a) $\int_e^\infty \frac{1}{x(\ln x)^p} dx.$

b) $\int_1^\infty \frac{dx}{\sqrt{x+x^3}}.$

c) $\int_1^\infty \frac{\sin x}{x^2+x+1} dx.$

d) $\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x^4}}.$

e) $\int_0^1 \frac{dx}{x-\sin x}.$

f) $\int_0^\infty \left(\sqrt[3]{x^3+1}-x\right) dx.$

g) $\int_0^\infty \frac{\sin x}{x} dx.$

h) $\int_0^\infty \frac{\cos x - \cos 3x}{x^2 \ln(1+\sqrt{x})} dx.$

Exercise 85. Find the value of the constant C for which the integral $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$ converges. Evaluate the integral for this value of C.

Exercise 86. Suppose f is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)dx$ is convergent?

8.1 - 8.3. Functions of Several Variables 1

Exercise 87. Find and sketch the domain of the function.

a) $f(x,y) = \sqrt{1-x^2} - \sqrt{4-y^2}.$

b) $f(x,y) = \arcsin(x^2+y^2-2).$

c) $f(x,y) = \frac{\sqrt{y-x^2}}{x^2-1}.$

d) $f(x,y) = \sqrt{x-y}\ln(x+y).$

Exercise 88. Find the domain and range of the function $f(x,y) = \sqrt{4-x^2-y^2}.$

Exercise 89. Find the limit, if it exits, or show that the limit does not exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}-1}$.

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2+y^4}$.

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin y}{3x^2+y^2}$.

d) $\displaystyle\lim_{(x,y)\to(\infty,\infty)} \frac{y^2}{x^2+3xy}$.

e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\left(e^{2y}-1\right)-2y\left(e^x-1\right)}{x^2+y^2}$.

f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3\cos y+y^3\cos x}{x^2+y^2}$.

Exercise 90. For what value of a is $f(x,y) = \begin{cases} x\arctan\dfrac{1}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ a, & \text{if } (x,y)=(0,0) \end{cases}$

continuous at every (x,y)?

Exercise 91. Find the first partial derivatives of the function.

a) $z=\sin\left(\dfrac{x}{1+xy}\right)$.

b) $z=(x^2+1)^y$.

c) $z=\int_y^{x^2}\sin\left(t^2\right)dt$.

d) $z=x^2\sin\dfrac{x}{y}$.

e) $z=\arctan\dfrac{x}{\sqrt{x^2+y^2}}$.

f) $u=x^2y\arcsin(y+z)$.

Exercise 92. Find $\partial z/\partial x$ and $\partial z/\partial y$.

a) $z=e^u\sin(uv)$, where $u=xy^2, v=x^2y$.

b) $z=\arcsin(u-v)$, where $u=x^2+y^2, v=1-2xy$.

8.4 - 8.6. Functions of Several Variables 2

Exercise 93. Find the second partial derivatives of the function.

a) $f(x,y)=\arctan\dfrac{y}{x}$.

b) $f(x,y)=\dfrac{xy}{x-y}$.

c) $f(x,y)=x\ln\left(x^2+y^2\right)$.

Exercise 94. Verify that the function $u(x,t)=\sin(x+at)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

Exercise 95. Show that the function $u = \sin x \cosh y + \cos x \sinh y$ is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Exercise 96. Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

a) Find $f_x(x,y)$ and $f_y(x,y)$.

b) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

Exercise 97. Find the linear approximation of the function $f(x,y) = \sqrt{20 - x^2 - 7y^3}$ at $(2,1)$ and use it to approximate $f(1.98, 1.05)$.

Exercise 98. Find the differential of the function.

a) $z = x^2 \ln(x + y^2)$.

b) $z = \arctan \dfrac{y}{x}$.

c) $z = xye^{xz}$.

d) $z = xy + \sinh(xy)$.

8.7 - 8.9. Functions of Several Variables 3

Exercise 99. Find the $d^2 f(x,y)$.

a) $f(x,y) = x^2y + y^2x, \quad (x,y) = (1,1)$.

b) $f(x,y) = \sin(xy)e^x, \quad (x,y) = (0,1)$.

c) $f(x,y,z) = x^2 + y^3 + z^4, \quad (x,y) = (-1,0,1)$.

d) $f(x,y,z) = \ln(1 + xyz), \quad (x,y,z) = (0,0,0)$.

Exercise 100. Use the Chain Rule to find dz/dt.

a) $z = \sqrt{1 + x^4 + y^2}, x = \ln t, y = \sin t$.

b) $z = \cos(x + 2y), x = 3t^2, y = 1/t$.

c) $z = xy + yz + xz, x = \sin t, y = \cos t, z = \tan t$.

d) $z = \dfrac{x+y}{y-z}, x = t^2, y = t^3, z = t^4$.

Exercise 101. Use the Chain Rule to find $\dfrac{\partial z}{\partial r}, \dfrac{\partial z}{\partial s}$.

a) $z = \cos(1 + xy), \quad x = r^2, y = s^2$.

b) $z = x^2y^3, \quad x = r\cos s, y = r\sin s$.

c) $z = e^x \sin y, \quad x = rs, y = r + s$.

d) $z = \tan\left(\dfrac{x}{y}\right), \quad x = r^2 + s^2, y = 2rs$.

Exercise 102. Use the Chain Rule to show that if $z = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$ then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

Exercise 103. Find the gradient of f.

a) $f(x, y) = e^x \sin y$.

b) $f(x, y) = \arctan(xy)$.

c) $f(x, y) = y^2e^{xy}$.

d) $f(x, y, z) = xe^y + ye^z + ze^x$.

Exercise 104. Find the directional derivative of f in the given direction $\mathbf{v}$.

a) $f(x, y) = e^y \cos x, \mathbf{v} = (1, 1)$.

b) $f(x, y) = \dfrac{x^2}{x^2 + y^2}, \mathbf{v} = (-1, 1)$.

c) $f(x, y) = \sin(x^2 + y^2), \mathbf{v} = (0, 2)$.

d) $f(x, y) = \sqrt{x^2 + y^2}, \mathbf{v} = (1, -1)$.

Exercise 105. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

a) $xy = \ln(y + z^2)$.

b) $x - z = \arctan(yz)$.

c) $\sin(xyz) = x + 2y + 3z^3$.

d) $x^3 + y^2 + z^3 + 6xyz = 1$.

e) $2x^2y + 4y^2 + x^2z + z^3 = 3$.

f) $xyz = \arcsin(x + y + z)$.

8.10 - 8.11. Functions of Several Variables 4

Exercise 106. Find the local maximum and minimum values and saddle point(s) of the function.

a) $z = xy^3 - 8x + 12y^2$.

b) $z = x^4 + y^4 - 4xy + 2$.

c) $z = 2x^3 + xy^2 + 5x^2 + y^2$.

d) $z = e^y (y^2 - x^2)$.

e) $z = e^{2x} (4x^2 - 2xy + y^2)$.

f) $z = x^4 + y^4 - x^2 - y^2 + 2xy$.

g) $z = x^2 + 4y^2 - 4xy + 2$.

h) $z = x^2 y e^{-x^2-y^2}$.

Exercise 107. Find the absolute maximum and minimum values of f on the set D.

a) $f(x, y) = x^4 + y^4 - 4xy + 2, D = \{(x, y) \mid 0 \le x \le 3; 0 \le y \le 2\}$.

b) $f(x, y) = xy^2, D = \{(x, y) \mid x \ge 0; y \ge 0; x^2 + y^2 \le 3\}$.

c) $f(x, y) = x^2 + y^2 + xy - 7x - 8y, D = \{(x, y) \mid x \ge 0; y \ge 0; x + y \le 6\}$.

Exercise 108. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Exercise 109. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

Exercise 110. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

a) $f(x, y) = 2x + 3y, x^2 + y^2 = 13$.

b) $f(x, y) = x^2 y, x^2 + 2y^2 = 6$.

c) $f(x, y) = e^{xy}, x^3 + y^3 = 16$.