Stability results for non-Newtonian fluid equations of Oldroyd-B type

Nguyen Thieu Huy
joint work with M. Geissert and M. Hieber

Hanoi University of Science and Technology

Mini-Workshop on PDE : AA

Oldroyd-B Equations

Consider Oldroyd-B equations on an exterior domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \mathrm{Re}(u_t + (u\cdot\nabla)u) - (1-\alpha)\Delta u + \nabla p = \mathrm{div}\tau + f & \text{in } \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty), \\ \mathrm{We}(\tau_t + (u\cdot\nabla)\tau + g_a(\tau,\nabla u)) + \tau = 2\alpha D(u) & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u\big|_{t=0} = u_0 & \text{in } \Omega, \\ \tau\big|_{t=0} = \tau_0 & \text{in } \Omega, \end{cases} \tag{1}$$

- u: Velocity
- τ: Purely elastic part of the stress tensor;
- Re , We: Reynolds, Weissenberg numbers
- $g_a(\tau,\nabla u) := \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u))$ for $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$- the deformation and vorticity tensors.

Intermezzo on Oldroyd-B model

Consider the momentum equation in incompressible case

$$u_t + (u \cdot \nabla)u = \mathrm{div}\sigma + f$$

Stokes' postulate: $\sigma = -pI + \tau$.

Intermezzo on Oldroyd-B model

Consider the momentum equation in incompressible case

$$u_t + (u \cdot \nabla)u = \mathrm{div}\sigma + f$$

Stokes' postulate: $\sigma = -pI + \tau$.

- Newtonian fluid: $\tau = 2\mu D(u)$.

Intermezzo on Oldroyd-B model

Consider the momentum equation in incompressible case

$$u_t + (u \cdot \nabla)u = \mathrm{div}\sigma + f$$

Stokes' postulate: $\sigma = -pI + \tau$.

- Newtonian fluid: $\tau = 2\mu D(u)$.
- Non-Newtonian fluid of Oldroyd-B type:

$$\tau + \lambda_1 \frac{\mathcal{D}_a \tau}{\mathcal{D}t} = 2\mu(D(u) + \lambda_2 \frac{\mathcal{D}_a D(u)}{\mathcal{D}t})$$

where $\frac{\mathcal{D}_a \tau}{\mathcal{D}t} = \left[\frac{\partial}{\partial t} + (u \cdot \nabla)\right] \tau + g_a(\tau, \nabla u)$
λ_1: relaxation time, λ_2: retardation time.

Intermezzo on Oldroyd-B model

Consider the momentum equation in incompressible case

$$u_t + (u \cdot \nabla)u = \mathrm{div}\sigma + f$$

Stokes' postulate: $\sigma = -pI + \tau$.

- Newtonian fluid: $\tau = 2\mu D(u)$.
- Non-Newtonian fluid of Oldroyd-B type:

$$\tau + \lambda_1 \frac{\mathcal{D}_a \tau}{\mathcal{D}t} = 2\mu(D(u) + \lambda_2 \frac{\mathcal{D}_a D(u)}{\mathcal{D}t})$$

where $\frac{\mathcal{D}_a \tau}{\mathcal{D}t} = \left[\frac{\partial}{\partial t} + (u \cdot \nabla)\right] \tau + g_a(\tau, \nabla u)$

λ_1: relaxation time, λ_2: retardation time.

Decomposing $\tau = \tau_N + \tau_E$ for $\tau_N = 2\mu \frac{\lambda_2}{\lambda_1} D(u)$, using again $\tau := \tau_E$ yield

$$\tau + \lambda_1 \frac{\mathcal{D}_a \tau}{\mathcal{D}t} = 2\mu\underbrace{(1 - \frac{\lambda_2}{\lambda_1})}_{\alpha} D(u).$$

Oldroyd-B Equations

Consider Oldroyd-B equations on an exterior domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \mathrm{Re}(u_t + (u \cdot \nabla)u) - (1-\alpha)\Delta u + \nabla p = \mathrm{div}\tau + f & \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ \mathrm{We}(\tau_t + (u \cdot \nabla)\tau + g_a(\tau, \nabla u)) + \tau = 2\alpha D(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u\big|_{t=0} = u_0 & \text{in } \Omega, \\ \tau\big|_{t=0} = \tau_0 & \text{in } \Omega, \end{cases} \tag{2}$$

- u: Velocity
- τ: Purely elastic part of the stress tensor;
- Re , We: Reynolds, Weissenberg numbers
- $g_a(\tau, \nabla u) := \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u))$ for $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$- the deformation and vorticity tensors.

Historical Remarks

Bounded domain Ω.

Historical Remarks

Bounded domain Ω.

• Guillopé and Saut (1990): Existence, uniqueness of solutions and exp. stability of stationary solutions in $L^2(\Omega)$ for small α.

Historical Remarks

Bounded domain Ω.

• Guillopé and Saut (1990): Existence, uniqueness of solutions and exp. stability of stationary solutions in $L^2(\Omega)$ for small α.

• Molinet and Talhouk (2004): Any $\alpha \in (0, 1)$.

Historical Remarks

Bounded domain Ω.

- Guillopé and Saut (1990): Existence, uniqueness of solutions and exp. stability of stationary solutions in $L^2(\Omega)$ for small α.
- Molinet and Talhouk (2004): Any $\alpha \in (0,1)$.

Whole space $\Omega = \mathbb{R}^3$.

- Lions and Masmoudi (2000): Existence of global weak solutions for $a = 0$, extended by Chemin, Masmoudi (2001); Lei, Liu, Zhou (2008).
- Kupfermann, Mangoubi and Titi (2008): Blow-up criteria for simplified models.
- Lei, Masmoudi and Zhou (2010): General case.

Historical Remarks

Bounded domain Ω.

• Guillopé and Saut (1990): Existence, uniqueness of solutions and exp. stability of stationary solutions in $L^2(\Omega)$ for small α.

• Molinet and Talhouk (2004): Any $\alpha \in (0, 1)$.

Whole space $\Omega = \mathbb{R}^3$.

• Lions and Masmoudi (2000): Existence of global weak solutions for $a = 0$, extended by Chemin, Masmoudi (2001); Lei, Liu, Zhou (2008).

• Kupfermann, Mangoubi and Titi (2008): Blow-up criteria for simplified models.

• Lei, Masmoudi and Zhou (2010): General case.

Exterior domain Ω.

• Hieber, Naito and Shibata (2012): Global existence and uniqueness of solutions in $L^2(\Omega)$ for small α.

• Fang, Hieber and Zi (2013): any $\alpha \in (0, 1)$.

• No stability result.

Linearized Problem

Consider the linearized problem of (2) (here set $\mathrm{Re} = \mathrm{We} = 1$):

$$\begin{cases} u_t - (1-\alpha)\Delta u + \nabla p &= \mathrm{div}\tau \quad \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \times (0,\infty), \\ \tau_t + \tau &= 2\alpha D(u) \quad \text{in } \Omega \times (0,\infty), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u\big|_{t=0} &= u_0 \quad \text{in } \Omega, \\ \tau\big|_{t=0} &= \tau_0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

Linearized Problem

Consider the linearized problem of (2) (here set $\mathrm{Re} = \mathrm{We} = 1$):

$$\begin{cases} u_t - (1-\alpha)\Delta u + \nabla p = \mathrm{div}\tau & \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ \tau_t + \tau = 2\alpha D(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u|_{t=0} = u_0 & \text{in } \Omega, \\ \tau|_{t=0} = \tau_0 & \text{in } \Omega, \end{cases} \tag{3}$$

Applying the Helmholtz projection $\mathbb{P}$, we have

$$\begin{pmatrix} \dot{u} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} (1-\alpha)\mathbb{P}\Delta & \mathbb{P}\mathrm{div} \\ 2\alpha D & -I \end{pmatrix} \begin{pmatrix} u \\ \tau \end{pmatrix}. \tag{4}$$

Oldroyd operator

Correspondingly, we consider the Oldroyd operator

$$\mathcal{B} := \begin{pmatrix} (1-\alpha)A_q & -\mathbb{P}\mathrm{div} \\ -2\alpha D & I \end{pmatrix} \tag{5}$$

acting on $L^q_\sigma(\Omega) \times W^{1,q}(\Omega)$ with the domain $(W^{2,q} \cap W^{1,q}_0 \cap L^q_\sigma) \times W^{1,q}$ where A_q is the Stokes operator $A_q := -\mathbb{P}\Delta$.

Existence and Strong Stability of Oldroyd Semigroups

Theorem

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary, $\mathcal{B}$ be the Oldroyd operator. Then,

(i) $-\mathcal{B}$ generates a bounded analytic semigroup $(e^{-t\mathcal{B}})_{t\geq 0}$ (Oldroyd semigroup) with the angle $\frac{\pi}{2} - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}$ on $L^q_\sigma(\Omega) \times W^{1,q}(\Omega)$ for all $1 < q < \infty$.

(ii) The semigroup $(e^{-t\mathcal{B}})_{t\geq 0}$ is strongly stable.

Sketch of the proof.

Sketch of the proof.

Step 1. We use the resolvent equation to prove

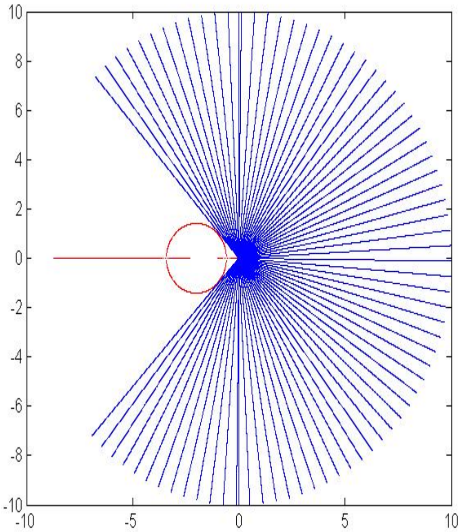
$$\lambda \in \sigma(-\mathcal{B}) \Leftrightarrow k(\lambda) = \frac{\lambda(\lambda+1)}{(1-\alpha)\lambda + 1 + \alpha} \in \sigma(-A_q).$$

Then, using $\sigma(-A_q) \subset (-\infty, 0]$, we obtain $\sigma(-\mathcal{B}) \subset \mathbb{C} \backslash \Sigma_{\pi - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}}$ where

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg(\lambda)| < \theta\}.$$

and we can compute $\sigma(-\mathcal{B})$ explicitly:

Picture



Proof

Step 2. We prove $k(\lambda) \in \Sigma_\pi \subset \rho(-A_q)$ for $\lambda \in \Sigma_{\pi - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}}$, then, prove the resolvent estimate

$$\left\|\lambda(\lambda+\mathcal{B})^{-1}\right\|_{L^q\times W^{1,q}} \leqslant M \text{ for all } \lambda \in \Sigma_{\pi - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}} \text{ and } 1 < q < \infty.$$

$\Longrightarrow -\mathcal{B}$ generates a bounded analytic semigroup $(e^{-t\mathcal{B}})_{t\geq 0}$.

Proof

Step 2. We prove $k(\lambda) \in \Sigma_\pi \subset \rho(-A_q)$ for $\lambda \in \Sigma_{\pi - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}}$,

then, prove the resolvent estimate

$$\left\|\lambda(\lambda + \mathcal{B})^{-1}\right\|_{L^q \times W^{1,q}} \leqslant M \text{ for all } \lambda \in \Sigma_{\pi - \arcsin\sqrt{\frac{2\alpha}{1+\alpha}}} \text{ and } 1 < q < \infty.$$

$\Longrightarrow -\mathcal{B}$ generates a bounded analytic semigroup $(e^{-t\mathcal{B}})_{t \geq 0}$.

Step 3. Using $\| - \mathcal{B}e^{-t\mathcal{B}}\| \leqslant \frac{M}{t},\ t > 0$, and $0 \notin \sigma_r(-\mathcal{B})$ we can prove the strong stability.

Oldroyd-B Equations without External Forces

Consider the Oldroyd-B Equation with $f = 0$.

$$
\begin{cases}
\mathrm{Re}(u_t + (u\cdot\nabla)u) - (1-\alpha)\Delta u + \nabla p &= \mathrm{div}\tau \quad \text{in } \Omega\times(0,\infty),\\
\nabla\cdot u &= 0 \quad \text{in } \Omega\times(0,\infty),\\
\mathrm{We}(\tau_t + (u\cdot\nabla)\tau + g_a(\tau,\nabla u)) + \tau &= 2\alpha D(u) \quad \text{in } \Omega\times(0,\infty),\\
u &= 0 \quad \text{on } \partial\Omega\times(0,\infty),\\
u\big|_{t=0} &= u_0 \quad \text{in } \Omega,\\
\tau\big|_{t=0} &= \tau_0 \quad \text{in } \Omega,
\end{cases}
\tag{6}
$$

- Hieber, Naito, Shibata (JDE 2012): Existence and uniqueness of the global solutions to (6) for small α.
- Hieber, Fang, Zhi (Math. Anal. 2013): Any $\alpha \in (0,1)$.

Theorem

Theorem

Denote $E_1 := H^3(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega)$. Then, solution $(0,0)$ is L^2-stable in the sense that every other solutions (u, τ) of (6) starting from a small ball centered at $(0,0)$ in $E_1 \times H^2(\Omega)$ satisfy

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = \lim_{t\to\infty} \|\tau(t)\|_{L^2} = 0.$$

Ideas for Proof

• Taking the inner product, using Schwarz's and Gronwall's inequalities to obtain

$$\|\tau(t)\| \leqslant e^{-Ct}\|\tau(0)\|+\sqrt{\frac{\alpha}{\mathrm{We}}}\int_0^t e^{-C(t-\xi)}\|D(u(\xi))\|d\xi \text{ for all } t \geq 0. \tag{7}$$

• Direct computations using Cauchy-Schwarz's inequalies lead to

$$\|u(t)\| \leqslant \frac{1}{t}\int_0^t \|u(s)\|ds + \frac{\sqrt{2/\mathrm{Re}}}{t^{1/2}}\left(\int_0^t \|\tau(s)\|\|\nabla u(s)\|ds\right)^{1/2}. \tag{8}$$

• Using the estimate of τ in (7) and Young inequality we obtain $\sup_{t\geq 0}\left(\int_0^t \|\tau(s)\|\|\nabla u(s)\|ds\right) < \infty$

• Using weak-L^p spaces, testing u with some relevant test function, and using time-regularity of $u(t)$ we obtain

$$\frac{1}{t}\int_0^t \|u(s)\|ds \leqslant \frac{1}{t}\int_0^t \|e^{-sA}u(0)\|ds + \frac{C_1}{t^{1/4}} + \frac{C_2}{t} \longrightarrow 0.$$

Further Research

- Stability of zero solution in L^p-spaces
- Existence and Stability of steady-state solutions when $f \neq 0$.
- Existence and of periodic solutions

Thank you for your attention!