Entropy- and Duality Methods for Dissipative PDE Models

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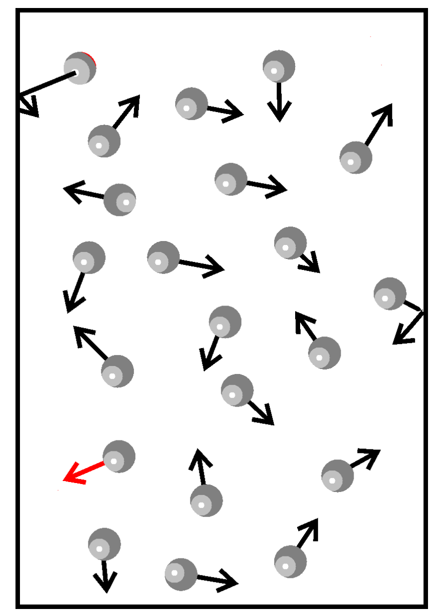
Entropy Methods in non-linear PDEs

Introduction

The Boltzmann's equation of gas dynamics describes the evolution of a phase-space density $f(t,x,v)$ (time $t \geq 0$, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$)

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)$$



Boltzmann's H-theorem: $H = \int_{\mathbb{R}^3} f \log(f)\, dv$

$$\frac{d}{dt} \int_{\mathbb{R}^3} H(f(t))\, dx \ = \int_{\mathbb{R}^3} Q(f,f) \log(f)\, dv \leq 0$$

Lyapunov functional $\Rightarrow$ irreversibility

Entropy Methods in non-linear PDEs

Overview

More precisely: Entropy and Entropy Dissipation functional[a]

$$\frac{d}{dt}\int_{\mathbb{R}^3} H(f(t))\,dx = -\int_{\mathbb{R}^3} D(f(t))\,dx \leq 0$$

How to use such an entropy structure in the best possible way in the quantitative analysis of

- Reaction-Diffusion Systems
- Coagulation-Fragmentation Models

[a] e.g. [DiPerna, Lions, 1988], [Desvillettes, Villani, 2005]

Reaction-Diffusion Systems

Prototypical quadratic system: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

- One reversible reaction of four species $\mathcal{A}_i$

$$\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$$

- mass action law kinetics: reaction rates $\sim a_1 a_2 - a_3 a_4$
- bounded, smooth domain $\Omega \subset \mathbb{R}^N$, $|\Omega| = 1$
  homogeneous Neumann boundary conditions

Reaction-Diffusion Systems

Prototypical quadratic system: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Concentrations $a_i(t,x)$ of $\mathcal{A}_i$, different diffusivities $d_i > 0$

$$\partial_t a_1 - d_1 \triangle_x a_1 = -a_1\, a_2 + a_3\, a_4$$
$$\partial_t a_2 - d_2 \triangle_x a_2 = -a_1\, a_2 + a_3\, a_4$$
$$\partial_t a_3 - d_3 \triangle_x a_3 = +a_1\, a_2 - a_3\, a_4$$
$$\partial_t a_4 - d_4 \triangle_x a_4 = +a_1\, a_2 - a_3\, a_4$$

- quadratic non-linearities, Bootstrap?
- no comparison principle, no invariant regions, Turing instability?
- NO, there is an entropy functional!

Entropy and Entropy Dissipation

Prototypical quadratic model: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Kinetic (free energy) entropy

$$H(a_i) = \int_\Omega \sum_{i=1}^{4} (a_i \ln(a_i) - a_i)\, dx$$

Entropy dissipation $\frac{d}{dt} H = -D \leq 0$

$$D(a_i) = 4 \sum_{i=1}^{4} \int_\Omega d_i |\nabla \sqrt{a_i}|^2 \, dx + \int_\Omega (a_1\, a_2 - a_3\, a_4) \ln \frac{a_1\, a_2}{a_3\, a_4}\, dx \geq 0$$

Programme

- Global (classical, weak, renormalised) solutions
- Large-time behaviour: explicit exponential convergence

Equilibrium and Entropy Dissipation

Prototypical quadratic model: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Equilibrium state $\{a_{i\infty}\}_{i=1..4}$ is the unique vector of positive constants balancing the reaction rate

$$a_{1\infty}\, a_{3\infty} = a_{2\infty}\, a_{4\infty},$$

and satisfying the three (linear indep.) mass-conservation laws (homogeneous Neumann boundary conditions)

$$a_{1\infty} + a_{2\infty} = \frac{1}{|\Omega|}\int_\Omega (a_{10} + a_{20})\, dx,$$

$$a_{1\infty} + a_{4\infty} = \frac{1}{|\Omega|}\int_\Omega (a_{10} + a_{40})\, dx,$$

$$a_{2\infty} + a_{3\infty} = \frac{1}{|\Omega|}\int_\Omega (a_{20} + a_{30})\, dx.$$

The Entropy Method

Quantitative large-time behaviour of dissipative PDEs

$E(f)$ non-increasing convex entropy functional

$D(f)$ entropy dissipation, f_∞ entropy minimising equilibrium

$$\frac{d}{dt}E(f) = \frac{d}{dt}(E(f) - E(f_\infty)) = -D(f) \leq 0$$

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provided conservation laws: $D(f) = 0 \Longleftrightarrow f = f_\infty$

$$D \geq \Phi(E(f) - E(f_\infty)), \quad \Phi(0) = 0, \quad \Phi \geq 0$$

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$\Rightarrow$ explicit convergence in entropy, exponential if $\Phi'(0) > 0$

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$\Rightarrow$ convergence in L_1 : $\|f - f_\infty\|_1^2 \leq C(E(f) - E(f_\infty))$

Cziszár-Kullback-Pinsker inequalities of convex entropies

My Personal Entropy Method Dictionary

- Understanding Entropy-Dissipation Structure $\Longleftrightarrow$ Entropy Entropy-Dissipation (EDD) Estimate

$$D \geq \Phi(E(f) - E(f_\infty)), \quad \Phi(0) = 0, \quad \Phi \geq 0$$

- Really Understanding ED Structure $\Longleftrightarrow$ Optimal Rate/Constant in EED Estimate

Entropy Entropy-Dissipation Estimate

$D \geq C\,(E - E_\infty)$

Theorem:[a] For any functions a_i, $i = 1, 2, 3, 4$ measurable, non-negative, satisfying the conservation laws holds

$$D(a_i) \geq C(M_{ij})(E(a_i|a_{i,\infty})).$$

Proof: Additivity $E(a_i|a_{i,\infty}) = E(a_i|\overline{a_i}) + E(\overline{a_i}|a_{i,\infty})$

$$E(a_i|\overline{a_i}) = \sum_{i=1}^{4} \int_\Omega a_i \ln\left(\frac{a_i}{\overline{a_i}}\right)\,dx \leq L(\Omega)\sum_{i=1}^{4}\int_\Omega |\nabla_x \sqrt{a_i}|^2\,dx\,,$$

+ Long long calculations + Conservation laws!

$\Rightarrow$ obtain Functional Inequality Not sharp!

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[a] [L. Desvillettes, K.F.]

Entropy Entropy-Dissipation Estimate

$D \geq C\,(E - E_\infty)$

Entopry Entropy-Dissipation Estimate

$$D(a_i) \geq C(M_{ij})(E(a_i|a_{i,\infty}))$$

+ Gronwall argument + Cziszár-Kullback inequality

$\Rightarrow$ explicit exponential convergence to equilibrium in L^1.

As long a suitable solutions exists?

Entropy A-priori Estimates

Entropy decay

The entropy decays

$$H(T) = H(0) - \int_0^T D(s)\,ds$$

with the entropy dissipation

$$D(a_i) = 4\sum_{i=1}^{4}\int_\Omega d_i |\nabla\sqrt{a_i}|^2\,dx + \int_\Omega (a_1\,a_2 - a_3\,a_4)\ln\frac{a_1\,a_2}{a_3\,a_4}\,dx$$

Entropy A-priori Estimates

Entropy decay

$$H(T) : \Big\{ \; a_i \in L^\infty([0,+\infty); L\log L(\Omega)) \qquad \forall i = 1,..,4$$

$$\int_0^T D(s) : \Big\{ \; \sqrt{a_i} \in L^2([0,+\infty); H^1(\Omega)) \qquad \forall i = 1,..,4 : d_i > 0$$

in 1D: $\|a_i\|^{3-\varepsilon}_{L^{3-\varepsilon}([0,T]\times[0,1])} \le C(1+T)$ + parabolic bootstrap

$\Rightarrow \; \|a_i\|_{L^\infty([0,T]\times[0,1])} \le C\big(1+T^{\frac{21}{2}}\big) \Rightarrow$ global classical solutions

in 2D: $a_i^2 \le a_i\, e^{\frac{s a_i}{\|\sqrt{a_i}(t)\|^2_{H^1(\Omega)}}} + \frac{a_i\|\sqrt{a_i}(t)\|^2_{H^1(\Omega)}}{s}\ln(a_i^2)$ for $\ln(a_i) > 1$

Trudinger ineq. $\Rightarrow$ global L^2 bound: $\|a_i\|^2_{L^2([0,T]\times\Omega)} \le C(1+T)$

$\Rightarrow$ global weak (super-)solutions [M. Pierre 2003][a]

[a] [Desvillettes F. Pierre Vovelle]

Entropy A-priori Estimates

Entropy decay

$$H(T) : \left\{ \; a_i \in L^\infty([0,+\infty); L \log L(\Omega)) \qquad \forall i = 1,..,4 \right.$$

$$\int_0^T D(s) : \begin{cases} \sqrt{a_i} \in L^2([0,+\infty); H^1(\Omega)) & \forall i = 1,..,4 : d_i > 0 \\ \int_0^T \int_\Omega (a_1\, a_2 - a_3\, a_4) \ln\left(\frac{a_1\, a_2}{a_3\, a_4}\right) dxdt \le C & \end{cases}$$

in 3+D: $\|a_i\|^{1+2/N}_{L^{1+2/N}([0,T]\times\Omega)} \le C(1+T)$

$\Rightarrow$ renormalised solutions (in the sense of [DiPerna, Lions]) [a]

[a] [Desvillettes F. Pierre Vovelle]

Duality Argument for Entropy Density

Entropy density equation: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Denote $z_i = a_i \ln(a_i) - a_i \quad \Rightarrow \quad H(a_i) = \int_\Omega \sum_{i=1}^4 z_i \, dx$

$$\begin{cases} \partial_t \left(\sum_{i=1}^4 z_i\right) - \Delta_x \left(\sum_{i=1}^4 d_i \, z_i\right) \leq 0, & t \in [0,T], \; x \in \Omega, \\ n \cdot \nabla_x z_i = 0, & t \in [0,T], \; x \in \partial\Omega, \end{cases}$$

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rewrite with $z := \sum_{i=1}^4 z_i$ and $M(t,x) := \frac{\sum_{i=1}^4 d_i \, z_i}{z}$ as

$$\begin{cases} \partial_t z - \Delta_x [M \, z] \leq 0, & t \in [0,T], \; x \in \Omega, \\ n \cdot \nabla_x [M \, z] = 0, & t \in [0,T], \; x \in \partial\Omega, \end{cases}$$

coefficient M bounded by $\min\{d_i\} \leq M(t,x) \leq \max\{d_i\}$!

Duality Argument for Entropy Density

A duality argument

Simplest case: $0 < d_0 \leq \min\{d_i\} \leq M(t,x) \leq \max\{d_i\} < \infty$[a]

$$\begin{cases} \partial_t z - \Delta_x [M\, z] \leq 0, & t \in [0,T],\ x \in \Omega, \\ n \cdot \nabla_x [M\, z] = 0, & t \in [0,T],\ x \in \partial\Omega, \end{cases}$$

Dual problem (inhomogeneous), $\quad Q_T = (0,T) \times \Omega$

$$-(w_t + M\, \Delta w) = \psi \in C_0^\infty(Q_T), \qquad w(T) = 0, \qquad n \cdot \nabla_x w \text{ on } \partial\Omega$$

one can show $\int_{Q_T} z\psi \leq \int_\Omega z(0) w(0) \leq C_T \|z(0)\|_{L^2} \|\psi\|_{L^2(Q_T)}$

$$\forall\, \psi \geq 0 \quad \Rightarrow \quad \|z\|_{L^2(Q_T)} \leq C_T \|z(0)\|_{L^2(\Omega)}$$

[a] [Pierre, Schmidt, 2000]

Duality Argument for Entropy Density

Entropy density equation: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Simplest case: $0 < d_0 \le \min\{d_i\} \le M(t,x) \le \max\{d_i\} < \infty$

$$\begin{cases} \partial_t \left(\sum_{i=1}^k z_i\right) - \Delta_x \left(\sum_{i=1}^k d_i\, z_i\right) \le 0, & t \in [0,T],\ x \in \Omega, \\ n \cdot \nabla_x z_i = 0, & t \in [0,T],\ x \in \partial\Omega, \end{cases}$$

uniform $L^2(\log L)^2$ bound + quadratic non-linearities

$\Rightarrow$ uniform integrability of non-linearities

$\Rightarrow$ convergence in $L^1(Q_T)$ of approximating sequence

$\Rightarrow$ global L^2-weak solutions in all space dimensions!![a]

[a] [Desvillettes, F., Pierre, Vovelle, Adv. Nonlinear Stud. 2007]

Duality Argument for Entropy Density

Entropy density equation: $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

Degenerate Diffusion: Enhanced entropy duality argument for

$$\begin{cases} \partial_t \left(\sum_{i=1}^k z_i\right) - \Delta_x \left(\sum_{i=1}^k d_i\, z_i\right) \le 0, & t \in [0,T],\ x \in \Omega, \\ n \cdot \nabla_x z_i = 0, & t \in [0,T],\ x \in \partial\Omega, \end{cases}$$

proves $(\sum_{i=1}^k z_i)\,(\sum_{i=1}^k d_i\, z_i) \in L^1([0,T] \times \Omega)$![a]

if $d_i(x) = 0$, but $\sum_{i=1}^4 d_i \ge d_0 > 0$ and $\nabla_x \sqrt{d_i} \in L^\infty(Q_T)$

$\Rightarrow$ still global weak solutions in all space dimensions!

[a] [Desvillettes, F., Pierre, Vovelle, Adv. Nonlinear Stud. 2007]

Duality Argument for Entropy Density

General systems $\partial_t a_i - \nabla_x \cdot (d_i \nabla_x a_i) = f_i(a)$

Quadratic Lotka-Volterra systems in $\mathbb{R}^N$: For $z \in (0,\infty)^q$

$$\begin{cases} \partial_t a_i = d_i \Delta_x a_i + a_i \sum_{j=1}^{q} p_{ij}(a_j - z_j), \quad i = 1 \ldots q \\ \nabla_x a_i \cdot n = 0 \quad \text{on} \quad \partial\Omega, \qquad a_i(0,\cdot) = a_{i,0}(\cdot) \in L^2(\Omega), \end{cases}$$

Then, there exists a global weak solution in $L^2(\Omega)$ in $\mathbb{R}^N$.

Improved Duality Argument

An improved duality argument

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega \in C^{2+\alpha}$. Let $T > 0$,

$$\begin{cases} \partial_t u - \Delta_x(M(t,x)u) = 0 & \text{on} \quad \Omega_T, \\ u(0,x) = u_0(x) \in L^p(\Omega) & \text{for} \quad x \in \Omega, \\ \nabla_x u \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega, \end{cases}$$

with $\quad 0 < a \le M(t,x) \le b < +\infty \quad$ for $\quad (t,x) \in \Omega_T$.

Then, any weak solution u satisfies $(1/p + 1/p' = 1)$

$$\|u\|_{L^p(\Omega_T)} \le (1 + b\, D_{a,b,p'})\, T^{1/p}\, \|u_0\|_{L^p(\Omega)}, \qquad p \in (2, +\infty),$$

where $\quad D_{a,b,p'} := \dfrac{C_{\frac{a+b}{2},p'}}{1 - C_{\frac{a+b}{2},p'} \frac{b-a}{2}} \quad$ as long as $\quad C_{\frac{a+b}{2},p'} \frac{b-a}{2} < 1$. [a]

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[a] [Cañizo Desvillettes F., CPDE]

Improved Duality Argument

Maximal regularity for dual problem

For $m > 0$ and $p' \in (1,2)$, the constant $C_{m,p'} > 0$ is the best parabolic regularity constant in

$$\|\Delta_x v\|_{L^{p'}(\Omega_T)} \le C_{m,p'} \, \|f\|_{L^{p'}(\Omega_T)},$$

where $v : [0,T] \times \Omega \to \mathbb{R}$ solves the backward heat equation

$$\begin{cases} \partial_t v + m\,\Delta_x v = f & \text{on} \quad \Omega_T, \\ v(T,x) = 0 & \text{for} \quad x \in \Omega, \\ \nabla_x v \cdot \nu(x) = 0 & \text{on} \quad [0,T] \times \partial\Omega. \end{cases}$$

Note that $C_{m,p'} = C_{m,p'}(\Omega, N) < \infty$ does not depend on T.
In particular $C_{m,2} \le \frac{1}{m}$.

Improved Duality Argument

Meyer's type argument

We take $m := (a+b)/2$ and rewrite

$$\partial_t v + m\,\Delta_x v = (m - M)\;\Delta_x v + f.$$

Then, from the above estimate, we get with $\|m - M\|_\infty \le \frac{b-a}{2}$

$$\|\Delta_x v\|_{L^{p'}(\Omega_T)} \le C_{m,p'} \left(\frac{b-a}{2} \,\|\Delta_x v\|_{L^{p'}(\Omega_T)} + \|f\|_{L^{p'}(\Omega_T)} \right).$$

Thus, $\|\Delta_x v\|_{L^{p'}(\Omega_T)} \le C\,\|f\|_{L^{p'}(\Omega_T)}$ provided $C_{m,p'}\,\frac{b-a}{2} < 1$.

Thus, $\|u\|_{L^p(\Omega_T)} \le (1 + b\,D_{a,b,p'})\,T^{1/p}\,\|u_0\|_{L^p(\Omega)}, \quad p \in (2, +\infty)$.

where $\quad D_{a,b,p'} := \frac{C_{\frac{a+b}{2},p'}}{1 - C_{\frac{a+b}{2},p'}\frac{b-a}{2}}$ $\quad$ provided $C_{\frac{a+b}{2},p'}\,\frac{b-a}{2} < 1$ [a]

[a] [Cañizo Desvillettes F., CPDE]

Summary

Results $\mathcal{A}_1 + \mathcal{A}_2 \leftrightarrow \mathcal{A}_3 + \mathcal{A}_4$

- 1D: global classical solutions based on entropy structure
  $\approx$ alternative to theory of [Amann],... (more general)
  explicit exponential decay (rates) in all Sobolev norms.
- 2D: global classical solutions [Goudon, Vasseur] [Cañizo Desvillettes F.]
  explicit exponential decay (rates) in L^2.
- allD: global weak L^2-solutions
  explicit exponential decay (rates) in L^p, $1 \leq p < 2$.
  Blow-up example [M. Pierre D. Schmidt]

Towards optimal constants/rates?

Nonlinear mass action law system $\alpha\mathcal{A} \leftrightarrow \beta\mathcal{B}$

$$\begin{aligned} \partial_t a - d_a \triangle a &= \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \cdot \begin{pmatrix} l\, a^\alpha \\ k\, b^\beta \end{pmatrix} \\ \partial_t b - d_b \triangle b &= \end{aligned}$$

Linearisation around equilibrium $u = a - a_\infty$ and $v = b - b_\infty$

$\Rightarrow$ Rescaled linearised system

$$\begin{aligned} \partial_t u - d_a \triangle u &= \begin{pmatrix} -\alpha^2 & \alpha\beta \\ \alpha\beta & -\beta^2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \\ \partial_t v - d_b \triangle v &= \end{aligned}$$

Fourier expansion $u = \sum_{k=0}^{\infty} u^k(t)\varphi_k$, $v = \sum_{k=0}^{\infty} v^k(t)\varphi_k$

$$\begin{cases} \Delta\varphi_k = \lambda_k\varphi_k & \text{in} \quad \Omega \\ n \cdot \nabla\varphi_k = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad k = 0, 1, \ldots$$

Towards optimal constants/rates?

Nonlinear mass action law system $\alpha\mathcal{A} \leftrightarrow \beta\mathcal{B}$

For any eigenmode $k \in \mathbb{N}$, we have

$$\partial_t \begin{pmatrix} u^k \\ v^k \end{pmatrix} = \begin{pmatrix} d_a\lambda_k - \alpha^2 & \alpha\beta \\ \alpha\beta & d_b\lambda_k - \beta^2 \end{pmatrix} \cdot \begin{pmatrix} u^k \\ v^k \end{pmatrix}, \qquad k = 0, 1, \ldots$$

a pair of eigenvalues μ_i, $i = 1, 2$ and eigenvectors e_i, $i = 1, 2$:

$$\mu_1(0) = 0, \qquad \mu_2(0) = -(\alpha^2 + \beta^2) < 0,$$

$$\mu_1(k) = \tfrac{d_a+d_b}{2}\lambda_k - \tfrac{\alpha^2+\beta^2}{2} + \sqrt{\tfrac{(\lambda_k(d_b-d_a)+\alpha^2-\beta^2)^2}{4} + \alpha^2\beta^2} < 0,$$

$$\mu_2(k) = \tfrac{d_a+d_b}{2}\lambda_k - \tfrac{\alpha^2+\beta^2}{2} - \sqrt{\tfrac{(\lambda_k(d_b-d_a)+\alpha^2-\beta^2)^2}{4} + \alpha^2\beta^2} < 0,$$

We have $\mu_2(k) < \mu_1(k)$ and $\mu_1(k+1) < \mu_1(k)$ for all $k \in \mathbb{N}$.

Towards optimal constants/rates?

Nonlinear mass action law system $\alpha\mathcal{A} \leftrightarrow \beta\mathcal{B}$

Two dominant negative eigenvalues

$$\mu_2(0) = \ -(\alpha^2 + \beta^2) < 0,$$

$$\mu_1(1) = \frac{d_a + d_b}{2}\lambda_k - \frac{\alpha^2 + \beta^2}{2} + \sqrt{\frac{(\lambda_k(d_b - d_a) + \alpha^2 - \beta^2)^2}{4} + \alpha^2\beta^2},$$

Special case: $d_1 = d_2 = d \quad \Rightarrow \quad \mu_1(1) = d\lambda_k$.

Optimal rate of convergence depends on

$$|\mu_1(1)| > |\mu_2(0)| \ \Leftrightarrow \ \pi^2 > \frac{\alpha^2}{d_2} + \frac{\beta^2}{d_1}$$

Geometry + stoichiometric coefficients + diffusion rates

Towards optimal constants/rates?

Nonlinear mass action law system $\alpha\mathcal{A} \leftrightarrow \beta\mathcal{B}$

Optimal constant in Entropy Entropy-Dissipation Estimate[a]

$$I = \min_{\substack{\beta\overline{u}+\alpha\overline{v}=0,\\ u,v\in C^\infty}} \left\{ \frac{d_a \int_\Omega |\nabla u|^2 + d_b \int_\Omega |\nabla v|^2 dx + \int_\Omega (\alpha u - \beta v)^2 dx}{\int_\Omega u^2 + v^2} \right\}$$

$$= \min_{c^0=0} \left\{ \frac{\sum_{k=0}^\infty |\mu_1(k)|(c^k)^2 + |\mu_2(k)|(d^k)^2}{\sum_{k=0}^\infty (c^k)^2 + (d^k)^2} \right\} \geq \min\{|\mu_1(1)|, |\mu_2(0)|\}$$

Minimising functions are $\sim \varphi_0$ (“large diffusion”) and $\sim \varphi_1$ (“small diffusion”)!?

Conjecture: nonlinear constants/minimisers are the same!

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[a] [G. Pissante, E. Latos, K. F.]

Coagulation-fragmentation models with diffusion

Background

The Formation and the Break-up of Clusters/Polymers

assume particles fully described by mass/size $y \in Y$.

full/realistic models can quickly get very difficult

Introduction

Coagulation-fragmentation models with diffusion

evolution of a polymer/cluster density $f(t,x,y) \geq 0$

$$\partial_t f - d(y)\triangle_x f = Q_{coag}(f,f) + Q_{frag}(f)$$

time $t \geq 0$, space $x \in \Omega$ normalised with $|\Omega| = 1$

size-dependent diffusion coefficients $d(y)$, y measures size

homogeneous Neumann B.C. $\nabla_x f(t,x,y) \cdot \nu(x) = 0$ on $\partial\Omega$

non-negative initial data $f_0(x,y) \geq 0$

Continuous coagulation-fragmentation models

inhomogeneous CF with constant kernel

Continuous in size density $f(t,x,y)$ with $y \in [0,\infty)$

$$\partial_t f - d(y)\triangle_x f = \int_0^y f(y-y')\,f(y')dy' - 2f(y)\int_0^\infty f(y')\,dy' + 2\int_y^\infty f(y')\,dy' - y\,f(y)$$

homogeneous Neumann, non-negative initial density $f_0(x,y)$

diffusion may degenerate at most linearly for large sizes

$$d(y) \le d^*(\delta)\,,\ \forall y \in [\delta,\delta^{-1}], \qquad 0 < \frac{d_*}{1+y} \le d(y)\,,\ \forall y \in [0,\infty).$$

[LM 2002] global existence of weak, solutions

inhomogeneous CF with constant kernel

Macroscopic densities

amount of monomers N, number density M

$$N = \int_0^\infty y' f(y')\,dy' \,, \quad M = \int_0^\infty f(y')\,dy'$$

conservation of the total mass

$$\partial_t N - \triangle_x \left(\int_0^\infty y' d(y') f(y')\,dy' \right) = 0$$

$$\partial_t M - \triangle_x \left(\int_0^\infty d(y') f(y')\,dy' \right) = N - M^2$$

inhomogeneous CF with constant kernel

Entropy (free energy functional)

Entropy

$$H(f)(t,x) = \int_0^\infty (f \ln f - f)\; dy \,,$$

Entropy dissipation

$$\begin{aligned}
\frac{d}{dt}\int_\Omega H(f)\,dx &= -D_H(f) \\
D_H(f) &= \int_\Omega \int_0^\infty d(y) \frac{|\nabla_x f|^2}{f}\,dy\,dx \\
&\quad + \int_\Omega \int_0^\infty\!\!\int_0^\infty (f'' - f\,f')\ln\left(\frac{f''}{f\,f'}\right) dy\,dy'dx
\end{aligned}$$

inhomogeneous CF with constant kernel

Inequality by [Aizenman, Bak]'79

$$\int_0^\infty \int_0^\infty (f(y+y') - f(y)f(y')) \ln\left(\frac{f(y+y')}{f(y)f(y')}\right) dy\, dy' \geq$$

$$M\, H(f|f_N) + 2(M - \sqrt{N})^2$$

Entropy dissipation

$$D_H(f) \geq \int_\Omega \int_0^\infty d(y) \frac{|\nabla_x f|^2}{f}\, dy\, dx$$
$$+ M\, H(f|f_N) + 2(M - \sqrt{N})^2$$

inhomogeneous CF with constant kernel

Local and global equilibria

Intermediate equilibria with the moments N and $M = \sqrt{N}$

$$f_N = e^{-\frac{1}{\sqrt{N}}y}$$

the global equilibrium

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}}$$

- is constant in x satisfying $M_\infty^2 = N_\infty$
- preserves the initial mass $N_\infty = \int_0^\infty N(x)\,dx$

inhomogeneous CF with constant kernel

relative entropy, additivity

relative entropy

$$H(f|g) = H(f) - H(g)$$

additivity

$$H(f|f_\infty) = H(f|f_N) + H(f_N|f_\infty)$$

f_N and f_∞ do not need to have the same L^1_y–norm, but nevertheless

$$\int_\Omega H(f_N|f_\infty)\,dx = 2\left(\sqrt{\int_\Omega N\,dx} - \int_\Omega \sqrt{N}\,dx\right) \geq 0$$

inhomogeneous CF with constant kernel

Theorem

Nonnegative initial data $(1 + y + \ln f_0)f_0 \in L^1((0,1) \times (0,\infty))$ with positive initial mass $\int_0^1 N_0(x)\,dx = N_\infty > 0$ on $\Omega = (0,1)$.
At most linearly degenerating diffusion coefficients.

Then, for $\beta < 2$ and $t > 0$

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1_{x,y}} \leq C_\beta\, e^{-(\ln t)^\beta},$$

and

$$\int_0^\infty (1+y)^q\, \|f(t,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\, dy \leq C_{\beta,q}\, e^{-(\ln t)^\beta}\,,$$

for all $t \geq t_* > 0$.

inhomogeneous CF with constant kernel

Entropy Entropy-Dissipation Estimate

Let $f := f(x,y) \geq 0$ be measurable with moments
$0 < \mathcal{M}_* \leq M(x) = \int_0^\infty f(x,y)\,dy \leq \|M\|_{L^\infty_x}$,
$0 < N_\infty = \int_\Omega \int_0^\infty y\,f(x,y)\,dydx$, $\int_\Omega \int_0^\infty y^{2p} f(x,y)\,dxdy \leq \mathcal{M}_{2p}$.

Then, for all $A \geq 1$ and $p > 1$

$$D_1(f) \geq \frac{C}{A\,\|M\|_{L^\infty_x}} \int_\Omega H(f|f_\infty)\,dx - C\,\frac{\mathcal{M}_{2p}}{A^{2p+1}}\,,$$

with a constant $C = C(\mathcal{M}_*, N_\infty, d_*, P(\Omega))$ depending only on $\mathcal{M}_*$, N_∞, d_*, and the Poincaré constant $P(\Omega)$.

Proof of Theorem

Algebraic rate for all $p > 1$

We have for any $A > 1$

$$\frac{d}{dt}\int_0^1 H(f|f_\infty)\,dx \le -\frac{C}{\|M\|_{L^\infty_x}}\frac{1}{A}\int_0^1 H(f|f_\infty)\,dx + \frac{C_p\,2^{8p^2}}{A^{2p+1}}\,,$$

where $\|M\|_{L^\infty_x}(t) \le m_\infty + m_2(t)$.

balance r.h.s. by choosing $A = A(t) > 2$ as

$$\frac{1}{A} \le C_3^{-1/2}\left(\frac{C_4\int_0^1 H(f|f_\infty)\,dx}{\|M\|_{L^\infty_x}\,2^{8p^2}}\right)^{\frac{1}{2p}}\,,$$

Thus, Gronwall yields algebraic rate for all $p > 1$.

Proof of Theorem

Algebraic rate for all $p > 1$

Faster than polynominal rate

Then, by summing w.r.t. $p \in \mathbb{N}$

$$\int_0^1 H(f(t)|f_\infty)\,dx \leq L(t - C_7),$$

where (for all $1 < \alpha < 2$)

$$L^{-1}(t) = \sum_{q \geq 1,\ \text{even}} \frac{t^q}{(C_{10}\,q)^q\,2^{2q^2}} = \sum_{q \geq 1,\ \text{even}} t^q\,e^{-2q^2 \ln 2 - q\ln(q\,C_{10})}$$

$$\geq C(\alpha)\,e^{[\ln^{2(\alpha-1)}(t)]}$$

for all t large enough and $1 < \alpha < 2$.

Proof of Entropy Entropy-Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 1) Additivity

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f|f_N)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

Proof of Entropy Entropy-Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 2) "Reacting" Moments N and M

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f|f_N)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

$$\sqrt{\overline{M_1}} - \overline{\sqrt{M_1}} \le \frac{2}{\sqrt{N_\infty}}\left[\|M - \sqrt{M_1}\|^2_{L^2_x} + \|M - \overline{M}\|^2_{L^2_x}\right].$$

Proof of Entropy Entropy-Dissipation Estimate

needs $\|M\|_{L^\infty_x}$**,** $\|M\|_{L^\infty_t(L^1_x)}$**,** $\mathcal{M}_* > 0$

Step 2) "Reacting" Moments N **and** $M > \mathcal{M}_* > 0$

$$\int_0^1 H(f|f_\infty)\,dx \le C\left[\int_0^1 M\,H(f|f_N)\,dx + 2\|M - \sqrt{M_1}\|^2_{L^2_x}\right] + \frac{4}{\sqrt{N_\infty}}\|M - \overline{M}\|^2_{L^2_x}$$

$$\le C\int_0^1\int_0^\infty\int_0^\infty (f'' - f\,f')\ln\left(\frac{f''}{f\,f'}\right)dy\,dy'\,dx + \frac{4}{\sqrt{N_\infty}}\|M - \overline{M}\|^2_{L^2_x}\,,$$

Proof of Entropy Entropy-Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 3) Diffusion For a cut-off size $A > 0$, denote

$M_A(t,x) := \int_0^A f(t,x,y)\,dy$ and $M_A^c(t,x) := \int_A^\infty f(t,x,y)\,dy$

$$\begin{aligned}\|M - \overline{M}\|^2_{L^2_x} &= \int_\Omega \left(M_A - \overline{M_A} + M_A^c - \overline{M_A^c}\right)^2 dx \\ &\le 2\|M_A - \overline{M_A}\|^2_{L^2_x} + \frac{4}{A^{2p}} \int_\Omega \left(\int_0^\infty y^p f(y)\,dy \right)^2 dx \\ &\le C(P, d_*)\, A\, \|M\|_{L^\infty_x} \int_\Omega \int_0^\infty d(y) \frac{|\nabla_x f|^2}{f}\, dy dx \\ &\quad + \frac{4}{A^{2p}} \|M\|_{L^\infty_x} \mathcal{M}_{2p}\end{aligned}$$

for any $p > 1$.

Proof of Entropy Entropy-Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Entropy Entropy Dissipation Estimate

Let $f := f(x,y) \geq 0$ be measurable with moments
$0 < \mathcal{M}_* \leq M(x) = \int_0^\infty f(x,y)\,dy \leq \|M\|_{L^\infty_x}$,
$0 < N_\infty = \int_\Omega \int_0^\infty y\, f(x,y)\,dydx$, $\int_\Omega \int_0^\infty y^{2p} f(x,y)\,dxdy \leq \mathcal{M}_{2p}$.

Then, for all $A \geq 1$ and $p > 1$

$$D_1(f) \geq \frac{C}{A\,\|M\|_{L^\infty_x}} \int_\Omega H(f|f_\infty)\,dx - C\,\frac{\mathcal{M}_{2p}}{A^{2p+1}}\,, \tag{1}$$

with a constant $C = C(\mathcal{M}_*, N_\infty, d_*, P(\Omega))$ depending only on $\mathcal{M}_*$, N_∞, d_*, and the Poincaré constant $P(\Omega)$.

Conclusions and Open Problems

Entropy- and Duality Methods for PDE models

- Good existence theory higher nonlinearities?
- Higher dimensions?
- Algebraic Structure of Reaction Networks?
- How to combine entropy and duality method?

THANK YOU VERY MUCH!!

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