Calculus 1 – Exercise sheets

Functions, essential functions

1. Determine the domain of

a) $y = \frac{x}{\sqrt{4x^2-1}}$ b) $y = \arcsin\frac{2x}{x+1}$ c) $y = \ln\frac{1-x}{1+x}$ d) $y = \sqrt{\arctan x}$.

2. Find the range of the functions

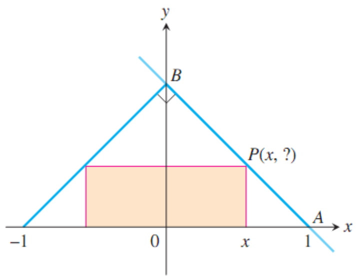
a) $y = \ln(1 - 2\cos x)$ b) $y = \arctan(e^x)$ c) $y = \sqrt{\arccos x}$ d) $y = \frac{x^2-1}{x^2+1}$.

3. As dry air moves upward, it expands and cools. The ground temperature is $30°C$ and the temperature at a height of $1\ km$ is $20°C$.

a) Express the temperature T (in $°C$) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

b) Draw the graph of the function. c) What is the temperature at a height of $4\ km$?

4. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



a) Express the y-coordinate of P in terms of x. b) Express the area of the rectangle in terms of x.

5. Determine whether f is even, odd, or neither

a) $f(x) = \frac{e^x - e^{-x}}{2} =: \sinh x$ b) $f(x) = \frac{2^x - x^2}{2^x + x^2}$ c) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$

d) $f(x) = \ln\frac{1-x}{1+x}$ e) $f(x) = \sin x + \cos x$.

6. Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains

a) $f(x) = \frac{1}{x+1}$ $g(x) = x - 1$ b) $f(x) = \sqrt{2x+3}$ $g(x) = x^2 + 1$.

c) $f(x) = \sin x$ $g(x) = \sqrt{x+1}$ d) $f(x) = 1 - x^2$ $g(x) = \frac{1-x}{1+x}$.

7. Find the inverse functions of

a) $y = 2\arcsin x$ b) $y = \frac{e^x - e^{-x}}{2}$ c) $y = \frac{1-x}{1+x}$ d) $y = \ln\frac{e^x - 1}{e^x + 1}$.

Limits and Continuity

8. Find the limit of the following sequences (if it exists)

a) $u_n = \sqrt[3]{n^3 + 2n^2} - n$ b) $u_n = \tan\left(\frac{2n\pi}{1+8n}\right)$ c) $u_n = \cos\left(\frac{n\pi}{2}\right)$

d) $u_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$ e) $u_n = \left(1 - \frac{1}{2n}\right)^n$ f) $u_n = \frac{n\cos(n^2+1)}{n^2+2}$.

9. Find the limit of the sequence $\{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots\}$.

10. Evaluate the limits

a) $\lim\limits_{x\to 1} \frac{\sqrt{x^2+8}-3}{x-1}$ b) $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3}$ c) $\lim\limits_{x\to 0} \frac{9^x - 5^x}{4^x - 3^x}$ d) $\lim\limits_{x\to 1} \frac{x^6-1}{x^{10}-1}$ e) $\lim\limits_{x\to 0} \frac{\ln(1+2x^2)}{1-\cos x}$

f) $\lim\limits_{x\to\infty} \left(\frac{x+1}{x+3}\right)^{2x}$ g) $\lim\limits_{x\to\infty} \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x$ h) $\lim\limits_{x\to 0} \frac{\ln(1+3\tan x)}{e^x - \cos x}$ i) $\lim\limits_{x\to 0^+} \frac{\sqrt{x^5}}{\sqrt{\sin x}\ln(1-3x^2)}$.

11. If $\lim\limits_{x\to 1} \frac{f(x)-8}{x-1} = 9$, find $\lim\limits_{x\to 1} f(x)$.

12. For what value of a is $f(x) = \begin{cases} x^2 + 2, & \text{if } x < 1 \\ 2ax^3 + 1, & \text{if } x \geq 1 \end{cases}$ continuous at every x?

13. Show that f is continuous on $(-\infty, \infty)$.

a) $f(x) = \begin{cases} \sin x & \text{if } x < \frac{\pi}{4}, \\ \cos x & \text{if } x \geq \frac{\pi}{4}. \end{cases}$ b) $f(x) = \begin{cases} x^2 & \text{if } x < 1, \\ \sqrt{x} & \text{if } x \geq 1. \end{cases}$

14. Locate the discontinuity of the function and illustrate by graphing

a) $y = \frac{1}{1+e^{1/x}}$ b) $y = \ln(\tan^2 x)$ c) $y = \frac{\sin x}{2^x - 1}$.

15. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither?

a) $f(x) = \begin{cases} \frac{2^x-1}{x} & \text{if } x < 0, \\ 2x + c & \text{if } x \geq 0. \end{cases}$ b) $f(x) = \begin{cases} \frac{\sin^2(\pi x)}{\ln(1+2x^2)} & \text{if } x < 1, \\ \sqrt{x} & \text{if } x \geq 1. \end{cases}$

16. Prove that there is a root of the given equation in the specified interval.

a) $x^6 - 3x + 1 = 0$, $(0, 1)$ b) $x^3 = \sqrt{3x+1}$, $(1, 2)$.

17. A train starts at 8AM from Hanoi to Haiphong, arriving at 11AM. The next day it starts at 8AM from Haiphong to Hanoi, arriving at 11AM. Is there a point on the route the train will cross at exactly the same time of day on both days?

18. Let f be a continuous function on a close interval $[0, 1]$ and $f(0) = 1, f(1) = 0$. Prove that there is a number $c \in (0, 1)$ at which $f(c) = c$.

Derivatives

19. Find the derivative of the function

a) $y = (x^2+1)\sqrt[3]{x^2+2}$ b) $y = \sin(\tan x)$ c) $y = \sqrt{x+\sqrt{x}}$

d) $y = \ln(x+\sqrt{x^2+5})$ e) $y = \sin^n x \cos nx$ f) $y = \left(1+\frac{1}{x}\right)^x$.

20. Find equations of the tangent line and the normal line to the curve

a) $y = \ln\left(x+\sqrt{x^2+3}\right)$ at $x = 1$ b) $y = x + \tanh(2x)$ at $x = 0$.

21. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

22. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a) perpendicular to the line $y = 1 - \frac{x}{24}$ b) parallel to the line $y = \sqrt{2} - 12x$.

23. Show that the tangents to the curve $y = \frac{\pi \sin x}{x}$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

24. For what values of a and b will

$$f(x) = \begin{cases} ax & if\ x < 2, \\ ax^2 + bx + 3 & if\ x \geq 2. \end{cases}$$

be differentiable for all values of x? Discuss the geometry of the resulting graph of f.

25. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$. Find the values of m and b that make f differentiable everywhere.

26. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.

27. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x) & if\ x \neq 0, \\ 0 & if\ x = 0, \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answer.

28. Find the *n*th derivatives of the function

a) $y = \frac{1}{x^2+x}$ b) $y = \frac{x}{x^2-4}$ c) $y = (x^2+1)e^{2x}$

d) $y = \ln(2x^2 + x)$ e) $y = (2x+1)\cos 3x$.

29. a) Given $f(x) = \ln\frac{x+1}{x+2}$, find $df(x)$, $d^{10}f(x)$.

b) Given $f(x) = (x+2)\ln x$, find $d^2f(1)$, $d^{20}f(1)$.

30. If f and g are differentiable functions with $f(0)=g(0)=0$ and $g'(0)\neq 0$, show that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{f'(0)}{g'(0)}.$$

31. Prove each of the following.

a) The derivative of an even function is an odd function.

b) The derivative of an odd function is an even function.

32. Find the derivative of the function $f(x)=\begin{cases} x\arctan\frac{1}{x^2} & if\ \ x\neq 0,\\ 0 & if\ \ x=0.\end{cases}$

33. Suppose that the functions f and g are defined throughout an open interval containing the points x_0, that f is differentiable at x_0, that $f(x_0)=0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0.

34. If $F(x)=f(3f(4f(x)))$, where $f(0)=0$ and $f'(0)=2$, find $F'(0)$.

Applications of derivatives

35. Evaluate the following limits

a) $\lim\limits_{x\to 0}\frac{\arcsin x-x}{x^3}$ b) $\lim\limits_{x\to 0}\frac{x^5-\ln(1+x^5)}{\sin^{10}x}$ c) $\lim\limits_{x\to 0}x\ln|x|$ d) $\lim\limits_{x\to+\infty}x[\pi-2\arctan(3x)]$

e) $\lim\limits_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$ f) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)$ g) $\lim\limits_{x\to-\infty}(x^2+2^x)^{\frac{1}{x}}$ h) $\lim\limits_{x\to 0}[\ln(e+2x)]^{\frac{1}{\sin x}}$

i) $\lim\limits_{x\to 0}[2x+e^{3x}]^{\frac{1}{\sin x}}$ j) $\lim\limits_{x\to 0^+}[\arcsin 2x]^{\tan x}$ k) $\lim\limits_{x\to 0}\frac{\sin x\ln(x+1)-x^2}{x^3}$.

36. Show that

a) $\sin(\arccos x)=\cos(\arcsin x)=\sqrt{1-x^2}$ for all $x\in[-1,1]$.

b) $\frac{1}{2}\arctan\frac{2x}{1-x^2}=\arctan x$ for all $x\in(-1,1)$. c) $\arcsin(\tanh x)=\arctan(\sinh x)$.

37. Prove that

a) $|\arcsin x-\arcsin y|\geq|x-y|$ for all $x,y\in[-1,1]$.

b) $\frac{y-x}{1+y^2}<\arctan y-\arctan x<\frac{y-x}{1+x^2}$ for all $0<x<y$.

c) $\frac{1}{2}-\frac{x}{8}<\frac{1}{x}-\frac{1}{e^x-1}<\frac{1}{2}$ for all $x>0$. d) $\frac{x}{x+1}\leq\ln(x+1)\leq x$ for all $x>-1$.

38. Show that the equation $3x+2\cos x+5=0$ has exactly one real root.

39. Show that the equation $a\cos x+b\cos 2x+c\cos 3x=0$ has at least one root on $(0,\pi)$.

40. Suppose that $f(x)$ is a continuous function on a close interval $[a,b]$ and differentiable on (a,b), and $f(a) = f(b) = 0$. Show that there exists a number $c \in (a,b)$ such that $f'(c) = 2021f(c)$.

41. For what values of a and b is the following equation true?

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0.$$

42. Determine the local extreme values

a) $y = x^{2/3}(x+2)$ b) $y = x^{2/3}(x^2-4)$ c) $y = x\sqrt{4-x^2}$ d) $y = x^2\ln x$

e) $y = (x^2-3)e^x$ f) $y = 3\arctan x - \ln(x^2+1)$ g) $y = \ln(x+3) + \operatorname{arccot} x$.

43. Find the absolute maximum and minimum values of $f(x) = x^2 + \frac{250}{x}$ over $[1, 10]$.

44. What value of a makes $f(x) = x^2 + \frac{a}{x}$ have

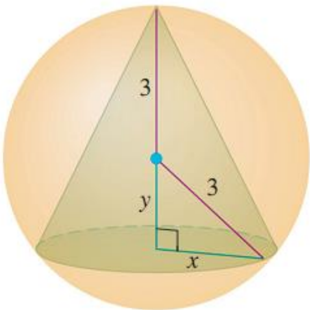
a) a local minimum at $x = 2$? b) a point of inflection at $x = 1$?

45. Find the intervals of concavity and the inflection points of

a) $f(x) = x^2\ln x$. b) $f(x) = (x+1)e^{-x}$ c) $f(x) = (1-x)\sqrt[3]{x}$.

46. (**The best fencing plan**) A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

47. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



48. (Designing a can) What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3.

49. A rectangle is to be inscribed under the arch of the curve $y = 4\cos(x/2)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

50. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

51. Find the *n*th-degree Taylor polynomials centered at $x = 0$ of $f(x)$. Determine the remainder.

a) $f(x) = x\cos x$, $n = 5$. b) $f(x) = \frac{x}{\sqrt{1+x^2}}$, $n = 5$ c) $f(x) = \sqrt{2+2x}$, $n = 3$.

52. Determine the asymptotes of the graph of $y = f(x)$

a) $y = \frac{e^x}{x+1}$ b) $y = x \operatorname{arccot} \frac{2}{x}$ c) $y = (x+2)e^{1/x}$ d) $y = \sqrt[3]{x^3 + x}$ e) $y = e^x \ln x$.

53. Determine the asymptotes of the curves

a) $x = t^3 - 3\pi,\ y = t^3 - 6 \arctan t$ b) $x = \frac{t^2}{t-1},\ y = \frac{t}{t^2-1}$.

54. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2+3x) - f(2+5x)}{x}.$$

55. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

56. What value of a and b make $f(x) = x^3 + ax^2 + bx$ have

a) a local maximum at $x = -1$ and a local minimum at $x = 3$?

b) a local minimum at $x = 4$ and a point of inflection at $x = 1$?

Integrations

57. Evaluate the following integrals

a) $\int \frac{x^3+1}{x^2+4} dx$ b) $\int \tan^4 x\, dx$ c) $\int \frac{1-2x}{\sqrt{2+x^2}} dx$ d) $\int \frac{x}{(x^2+1)(x+2)} dx$

e) $\int \frac{\sin 2x}{\sqrt{\sin^4 x+1}} dx$ f) $\int \frac{dx}{3\sin x - 4\cos x}$ g) $\int \frac{dx}{1+\sqrt{x^2+4x+5}}$ h) $\int \frac{x+1}{\sqrt{x^2-2x-1}} dx$.

58. Evaluate the following integrals

a) $\int (x+1) \arctan x\, dx$ b) $\int (x+2) \ln x\, dx$ c) $\int \arcsin^2 x\, dx$

d) $\int \frac{\arctan x}{x^2} dx$ e) $\int \frac{x}{(x^2+2x+2)^2} dx$ f) $\int \frac{e^{2x}}{1+e^x} dx$ g) $\int \sqrt{\frac{x}{x-1}} dx$

h) $\int \frac{x^2+2}{x^3-1} dx$ i) $\int \frac{x^2+1}{x^4+1} dx$ j) $\int \frac{\sin^2 x}{\cos^3 x} dx$ k) $\int \frac{1}{x^2\sqrt{x^2+1}} dx$.

59. Find a function $f(x)$ such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of $f(x)$.

60. A car is traveling at 100km/h when the driver sees an accident 80m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a pileup?

61. Show that a) $\frac{13}{42} < \int_0^1 \sin(x^2)\, dx < \frac{1}{3}$ b) $\int_0^{\pi/6} \cos(x^2)\, dx > \frac{1}{2}$.

62. Find the derivative of the following functions

a) $f(x) = \int_0^x \sqrt{1+t^4} dt$ b) $g(x) = \int_0^{x^2} \sin(t^2)\, dt$ c) $h(x) = \int_{x^3}^0 \sin^3 t\, dt$.

63. Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on $[0,1]$.

a) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5}$ b) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \cdots + \sqrt{1+\frac{n}{n}}\right)$.

Answer a) $1/5$ b) $\frac{2}{3}\left(2\sqrt{2}-1\right)$

64. Evaluate the following integrals

a) $\int_1^e (x\ln x)^2 dx$ b) $\int_0^{\pi/4} \frac{\sin^2 x \cos x}{(1+\tan^2 x)^2} dx$ c) $\int_1^2 \frac{\sqrt{x^2-1}}{x^2} dx$ d) $\int_0^1 \frac{\ln(x^2+1)}{(x+1)^2} dx$.

Answer a) $\frac{1}{27}(5e^3-2)$ b) $\frac{71\sqrt{2}}{1680}$ c) $\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$ d) $\frac{\pi}{4} - \ln 2$

65. If f is continuous on $[0,1]$, show that

a) $\int_0^{\pi/2} f(\sin x) dx = \int_0^{\pi/2} f(\cos x) dx$ b) $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$.

Evaluate c) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$ d) $\int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}} dx$.

Answer c) $\frac{\pi^2}{4}$ d) $\frac{\pi}{4}$

66. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a) $\int_0^{\infty} \frac{x}{(x^2+2)^2} dx$ b) $\int_1^{\infty} \frac{x+2}{x^2+3x} dx$ c) $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$ d) $\int_{-\infty}^{0} xe^{-x} dx$ e) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$

f) $\int_0^{\infty} \frac{e^x}{e^{2x}+3} dx$ g) $\int_0^{\infty} \frac{x\arctan x}{(1+x^2)^2} dx$ h) $\int_1^{\infty} \frac{x+1}{\sqrt{x^4-x}} dx$ i) $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$.

Answer a) $\frac{1}{4}$ b) divergent c) divergent d) divergent e) -4

f) $\frac{\pi}{3\sqrt{3}}$ g) $\frac{\pi}{8}$ h) divergent i) π.

67. Determine whether the improper integral is convergent or divergent.

a) $\int_e^{\infty} \frac{1}{x(\ln x)^p} dx$ b) $\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}$ c) $\int_1^{\infty} \frac{\sin x}{x^2+x+1} dx$ d) $\int_0^1 \frac{\sqrt{x}dx}{\sqrt{1-x^4}}$

e) $\int_0^1 \frac{dx}{x-\sin x}$ f) $\int_0^{\infty} \left(\sqrt[3]{x^3+1} - x\right) dx$ g) $\int_0^{\infty} \frac{\sin x}{x} dx$ h) $\int_0^{\infty} \frac{\cos x - \cos 3x}{x^2 \ln(1+\sqrt{x})} dx$.

Answer a) Convergent if $p > 1$, divergent if $p \le 1$. b) convergent c) convergent

d) convergent e) divergent f) convergent g) convergent h) convergent

68. Find the value of the constant C for which the integral $\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx$ converges. Evaluate the integral for this value of C.

68 Answer $C = 3$, $\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{3}{3x+1}\right) dx = -\ln 3$.

69. Suppose f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)dx$ is convergent?

70. Find the area of the region enclosed by the parabolas $x = 2y - y^2$, $x = y^2 - 4y$.

Answer 9

71. Find the area of the region enclosed by the curve $y^2 = x^2 - x^4$.

Answer $\frac{4}{3}$

72. Find the area of the region enclosed by $y = \frac{1}{x}$, $y = x$ and $y = \frac{1}{4}x$, $x > 0$.

73. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

74. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

a) $y = 2x - x^2$, $y = 0$; about the x-axis b) $y = \ln x$, $y = 1$, $y = 2$, $x = 0$; about the y-axis

c) $x = y^2$, $x = 1$; about $x = 1$ d) $y = x^2$, $x = y^2$; about $y = -1$.

Answer a) $\frac{16}{15}\pi$ b) $\frac{\pi}{2}(e^4 - e^2)$ c) $\frac{16}{15}\pi$ d) $\frac{29}{30}\pi$

75. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about

a) the x-axis b) the y-axis c) the line $x = 5$.

76. Find the length of the curves

a) $y = \frac{x^2}{8} - \ln x$, $4 \le x \le 8$ b) $x = y^{2/3}$, $1 \le y \le 8$

c) $x = 5\cos t - \cos 5t$, $y = 5\sin t - \sin 5t$, $0 \le t \le \pi/2$.

77. Find the area of the surface generated by revolving the curve

a) $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the x-axis

b) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \le x \le 2$, about the y-axis.

Functions of several variables

78. Find and sketch the domain of the function

a) $f(x,y)=\sqrt{1-x^2}-\sqrt{4-y^2}$ b) $f(x,y)=\arcsin(x^2+y^2-2)$

c) $f(x,y)=\frac{\sqrt{y-x^2}}{x^2-1}$ d) $f(x,y)=\sqrt{x-y}\ln(x+y)$.

79. Find the domain and range of the function $f(x,y)=\sqrt{4-x^2-y^2}$.

80. Find the limit, if it exits, or show that the limit does not exist

a) $\lim\limits_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{1+x^2+y^2}-1}$ b) $\lim\limits_{(x,y)\to(0,0)}\frac{x^2}{2x^2+y^4}$ c) $\lim\limits_{(x,y)\to(0,0)}\frac{x^2\sin y}{3x^2+y^2}$

d) $\lim\limits_{(x,y)\to(\infty,\infty)}\frac{y^2}{x^2+3xy}$ e) $\lim\limits_{(x,y)\to(0,0)}\frac{x(e^{2y}-1)-2y(e^x-1)}{x^2+y^2}$.

81. For what value of a is $f(x,y)=\begin{cases}x\arctan\frac{1}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ a, & \text{if } (x,y)=(0,0)\end{cases}$ continuous at every (x,y)?

82. Find the first partial derivatives of the function

a) $z=\sin\left(\frac{x}{1+xy}\right)$ b) $z=(x^2+1)^y$ c) $z=\int_y^{x^2}\sin(t^2)\,dt$

d) $z=x^2\sin\frac{x}{y}$ e) $z=\arctan\frac{x}{\sqrt{x^2+y^2}}$ f) $u=x^2y\arcsin(y+z)$.

83. Find $\partial z/\partial x$ and $\partial z/\partial y$

a) $z=e^u\sin(uv)$, where $u=xy^2$, $v=x^2y$

b) $z=\arcsin(u-v)$, where $u=x^2+y^2$, $v=1-2xy$.

84. Use the Chain Rule to find dz/dt if

a) $z=\sqrt{1+x^4+y^2}$, $x=\ln t$, $y=\sin t$ b) $z=\cos(x+2y)$, $x=3t^2$, $y=1/t$.

85. Find $y'(x)$ if y is defined implicitly as a function of x by the equation

a) $\arctan(2x+y)=y^3$ b) $x^3+y^3=3x^2y$ c) $\cos(x-y)=xe^y$.

86. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

a) $xy=\ln(y+z^2)$ b) $x-z=\arctan(yz)$ c) $\sin(xyz)=x+2y+3z^3$

d) $x^3+y^2+z^3+6xyz=1$ e) $2x^2y+4y^2+x^2z+z^3=3$.

87. Find the second partial derivatives of the function

a) $f(x,y)=\arctan\frac{y}{x}$ b) $f(x,y)=\frac{xy}{x-y}$ c) $f(x,y)=x\ln(x^2+y^2)$.

88. Verify that the function $u(x,t) = \sin(x + at)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

89. Show that the function $u = \sin x \cosh y + \cos x \sinh y$ is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

90. Let $f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

a) Find $f_x(x,y)$ and $f_y(x,y)$ b) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

91. Find the linear approximation of the function $f(x,y) = \sqrt{20 - x^2 - 7y^3}$ at $(2,1)$ and use it to approximate $f(1.98, 1.05)$.

92. Find the differential of the function

a) $z = x^2 \ln(x + y^2)$ b) $z = \arctan\frac{y}{x}$ c) $u = xye^{xz}$.

93. Find the local maximum and minimum values and saddle point(s) of the function

a) $z = xy^3 - 8x + 12y^2$ b) $z = x^4 + y^4 - 4xy + 2$ c) $z = 2x^3 + xy^2 + 5x^2 + y^2$

d) $z = e^y(y^2 - x^2)$ e) $z = e^{2x}(4x^2 - 2xy + y^2)$ f) $z = x^4 + y^4 - x^2 - y^2 + 2xy$

g) $z = x^2 + 4y^2 - 4xy + 2$ h) $z = x^2ye^{-x^2-y^2}$.

94. Find the absolute maximum and minimum values of f on the set D

a) $f(x,y) = x^4 + y^4 - 4xy + 2$, $D = \{(x,y) | 0 \leq x \leq 3;\ 0 \leq y \leq 2\}$

b) $f(x,y) = xy^2$, $D = \{(x,y) | x \geq 0;\ y \geq 0; x^2 + y^2 \leq 3\}$

c) $f(x,y) = x^2 + y^2 + xy - 7x - 8y$, $D = \{(x,y) | x \geq 0;\ y \geq 0; x + y \leq 6\}$

95. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

96. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

97. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint

a) $f(x,y) = 2x + 3y,\ x^2 + y^2 = 13$ b) $f(x,y) = x^2y,\ x^2 + 2y^2 = 6$

c) $f(x,y) = e^{xy},\ x^3 + y^3 = 16$.