**HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY**
FACULTY OF APPLIED MATHEMATICS AND INFORMATICS
ADVANCED TRAINING PROGRAM

## Lecture on

# INFINITE SERIES AND DIFFERENTIAL EQUATIONS

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**Ha Noi-2009**

# Preface

The *Lecture on infinite series and differential equations* is written for students of Advanced Training Programs of Mechatronics (from California State University–CSU Chico) and Material Science (from University of Illinois- UIUC). To prepare for the manuscript of this lecture, we have to combine not only the two syllabuses of two courses on Differential Equations (Math 260 of CSU Chico and Math 385 of UIUC), but also the part of infinite series that should have been given in Calculus I and II according to the syllabuses of the CSU and UIUC (the Faculty of Applied Mathematics and Informatics of HUT decided to integrate the knowledge of infinite series with the differential equations in the same syllabus). Therefore, this lecture provides the most important modules of knowledge which are given in all syllabuses.

This lecture is intended for engineering students and others who require a working knowledge of differential equations and series; included are technique and applications of differential equations and infinite series. Since many physical laws and relations appear mathematically in the form of differential equations, such equations are of fundamental importance in engineering mathematics. Therefore, the main objective of this course is to help students to be familiar with various physical and geometrical problems that lead to differential equations and to provide students with the most important standard methods for solving such equations.

I would like to thank Dr. Tran Xuan Tiep for his reading and reviewing of the manuscript. I would like to express my love and gratefulness to my wife Dr. Vu Thi Ngoc Ha for her constant support and inspiration during the preparation of the lecture.

Hanoi, April 4, 2009

Dr. Nguyen Thieu Huy

# Content

# CHAPTER 1: INFINITE SERIES

The early developers of the calculus, including Newton and Leibniz, were well aware of the importance of infinite series. The values of many functions such as sine and cosine were geometrically obtainable only in special cases. Infinite series provided a way of developing extensive tables of values for them.

This chapter begins with a statement of what is meant by infinite series, then the question of when these sums can be assigned values is addressed. Much information can be obtained by exploring infinite sums of constant terms; however, the eventual objective in analysis is to introduce series that depend on variables. This presents the possibility of representing functions by series. Afterward, the question of how continuity, differentiability, and integrability play a role can be examined.

The question of dividing a line segment into infinitesimal parts has stimulated the imaginations of philosophers for a very long time. In a corruption of a paradox introduce by Zeno of Elea (in the fifth century B.C.) a dimensionless frog sits on the end of a one-dimensional log of unit length. The frog jumps halfway, and then halfway and halfway ad infinitum. The question is whether the frog ever reaches the other end. Mathematically, an unending sum,

$$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots$$

is suggested. "Common sense" tells us that the sum must approach one even though that value is never attained. We can form sequences of partial sums

$$S_1=\frac{1}{2}, S_2=\frac{1}{2}+\frac{1}{4}, \ldots, S_n=\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}, \ \cdots$$

and then examine the limit. This returns us to Calculus I and the modern manner of thinking about the infinitesimal.
In this chapter, consideration of such sums launches us on the road to the theory of infinite series.

## 1. Definitions of Infinite Series and Fundamental Facts

**1.1 Definitions.** Let $\{u_n\}$ be a sequence of real numbers. Then, the formal sum

$$S=\sum_{n=1}^{\infty}u_n=u_1+u_2+\cdots+u_n+\cdots \qquad (1)$$

is an **infinite series**.

Its value, if one exists, is the limit of the sequence of **partial sums** $\{S_n=\sum_{k=1}^{n}u_k\}_{n=1}^{\infty}$

$$S=\lim_{n\to\infty} S_n$$

If the limit exists, the series is said to converge to that sum, *S*. If the limit does not exist, the series is said to diverge.
Sometimes the character of a series is obvious. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

generated by the frog on the log surely converges, while $\sum_{n=1}^{\infty} n$ diverges. On the other hand, the variable series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

raises questions.

This series may be obtained by carrying out the division 1/(1-x) . If -1 < x < 1, the sums $S_n$ yields an approximations to 1/(1-x), passing to the limit, it is the exact value. The indecision arises for x = -1. Some very great mathematicians, including Leonard Euler, thought that S should be equal to 1/2, as is obtained by
substituting -1 into 1/(1-x). The problem with this conclusion arises with examination of
1 -1 + 1 -1+ 1 -1 + • • • and observation that appropriate associations can produce values of 1 or 0. Imposition of the condition of uniqueness for convergence put this series in the category of divergent and eliminated such possibility of ambiguity in other cases.

### 1.2 Fundamental facts:

1. If $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n$ =0. The converse, however, is not necessarily true, i.e., if $\lim_{n\to\infty} u_n$ =0, $\sum_{n=1}^{\infty} u_n$ may or may not converge. It follows that if the $n^{th}$ term of a series does not approach zero, the series is divergent.

2. Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence.

3. Removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence or divergence.

### 1.3 Special series:

1. **Geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$, where $a$ and $r$ are constants, converges to $S = \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geqq 1$. The sum of the first $n$ terms is $S_n = \frac{a(1-r^n)}{1-r}$

2. **The *p* series** $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$, where $p$ is a constant, converges for $p > 1$ and diverges for $p \leqq 1$. The series with $p = 1$ is called the *harmonic series*.

## 2. Tests for Convergence and Divergence of Series of Constants

More often than not, exact values of infinite series cannot be obtained. Thus, the search turns toward information about the series. In particular, its convergence or divergence comes in question. The following tests aid in discovering this information.

### 2.1 Comparison test for series of non-negative terms.

(a) *Convergence.* Let $v_n \geqq 0$ for all $n > N$ and suppose that $\Sigma v_n$ converges. Then if $0 \leqq u_n \leqq v_n$ for all $n > N$, $\Sigma u_n$ also converges. Note that $n > N$ means *from some term onward.* Often, $N = 1$.

**EXAMPLE.** Since $\frac{1}{2^n+1} \leqq \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$ converges, $\sum \frac{1}{2^n+1}$ also converges.

(b) *Divergence.* Let $v_n \geqq 0$ for all $n > N$ and suppose that $\Sigma v_n$ diverges. Then if $u_n \geqq v_n$ for all $n > N$, $\Sigma u_n$ also diverges.

**EXAMPLE.** Since $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges.

PROOF OF COMPARISON TEST:

(a) Let $0 \le u_m \le v_n$, n = 1, 2, 3,... and $\sum_{n=1}^{\infty} v_n$ converges. Then, let $S_n = u_1 + u_2 + \ldots + u_n$; $T_n = v_1 + v_2 + \ldots + v_n$.

Since $\sum_{n=1}^{\infty} v_n$ converges, $\lim_{n\to\infty} T_n$ exists and equals T, say. Also, since $v_n \ge 0$, $T_n \le T$.

Then $S_n = u_1 + u_2 + \cdots + u_n \le v_1 + v_2 + \cdots + v_n \le T$ or $0 \le S_n \le T$.

Thus $\{S_n\}$ is a bounded monotonic increasing sequence and must have a limit, i.e., $\sum_{n=1}^{\infty} u_n$ converges.

(b) The proof of (b) is left for the reader as an exercise.

### 2.2 The Limit-Comparison or Quotient Test for series of non-negative terms.

(a) If $u_n \geqq 0$ and $v_n \geqq 0$ and if $\lim_{n\to\infty} \frac{u_n}{v_n} = A \neq 0$ and $A \neq \infty$ then $\Sigma u_n$ and $\Sigma v_n$ either both converge or both diverge.

(b) If $A = 0$ in (a) and $\Sigma v_n$ converges, then $\Sigma u_n$ converges.

(c) If $A = \infty$ in (a) and $\Sigma v_n$ diverges, then $\Sigma u_n$ diverges.

PROOF: (a)

By hypothesis, given $\epsilon > 0$ we can choose an integer $N$ such that $\left|\frac{u_n}{v_n} - A\right| < \epsilon$ for all $n > N$. Then for $n = N+1, N+2, \ldots$

$$-\epsilon < \frac{u_n}{v_n} - A < \epsilon \qquad \text{or} \qquad (A-\epsilon)v_n < u_n < (A+\epsilon)v_n \tag{1}$$

Summing from $N+1$ to $\infty$ (more precisely from $N+1$ to $M$ and then letting $M \to \infty$),

$$(A-\epsilon)\sum_{N+1}^{\infty} v_n \leqq \sum_{N+1}^{\infty} u_n \leqq (A+\epsilon)\sum_{N+1}^{\infty} v_n \tag{2}$$

There is no loss in generality in assuming $A - \epsilon > 0$. Then from the right-hand inequality of (2), $\Sigma u_n$ converges when $\Sigma v_n$ does. From the left-hand inequality of (2), $\Sigma u_n$ diverges when $\Sigma v_n$ does. For the cases A=0 or A=∞, it is easy to prove the assertions (b) and (c).

**EXAMPLE:** $\sum_{n=1}^{\infty}\sin\frac{1}{2^n}$ converges, since $\sin\frac{1}{2^n}>0$, $\lim_{n\to\infty}\frac{\sin\frac{1}{2^n}}{\frac{1}{2^n}}=1$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $v_n = 1/n^p$, we have the following theorem

***Theorem 1.*** Let $\lim_{n\to\infty} n^p u_n = A$. Then

(*i*) $\Sigma u_n$ converges if $p > 1$ and $A$ is finite.

(*ii*) $\Sigma u_n$ diverges if $p \leqq 1$ and $A \neq 0$ ($A$ may be infinite).

**EXAMPLES.** 1. $\sum \frac{n}{4n^3-2}$ converges since $\lim_{n\to\infty} n^2 \cdot \frac{n}{4n^3-2} = \frac{1}{4}$.

2. $\sum \frac{\ln n}{\sqrt{n+1}}$ diverges since $\lim_{n\to\infty} n^{1/2} \cdot \frac{\ln n}{(n+1)^{1/2}} = \infty$.

### 2.3 Integral test for series of non-negative terms.

If $f(x)$ is positive, continuous, and monotonic decreasing for $x \geqq N$ and is such that $f(n) = u_n, n = N, N+1, N+2, \ldots,$ then $\Sigma u_n$ converges or diverges according as $\int_N^\infty f(x)\,dx = \lim_{M\to\infty}\int_N^M f(x)\,dx$ converges or diverges. In particular we may have $N = 1$, as is often true in practice.

This theorem borrows from the next chapter since the integral has an unbounded upper limit. (It is an improper integral. The convergence or divergence of these integrals is defined in much the same way as for infinite series.)

**EXAMPLE:** $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges since $\lim_{M\to\infty}\int_1^M \frac{dx}{x^2} = \lim_{M\to\infty}\left(1-\frac{1}{M}\right)$ exists.

PROOF OF INTEGRAL TEST:

We perform the proof taking $N = 1$. Modifications are easily made if $N > 1$.
From the monotonicity of $f(x)$, we have

$$u_{n+1} = f(n+1) \leqq f(x) \leqq f(n) = u_n \qquad n = 1, 2, 3, \ldots$$

Integrating from $x = n$ to $x = n+1$, using Property 7, Page 92,

$$u_{n+1} \leqq \int_n^{n+1} f(x)\,dx \leqq u_n \qquad n = 1, 2, 3 \ldots$$

Summing from $n = 1$ to $M - 1$,

$$u_2 + u_3 + \cdots + u_M \leqq \int_1^M f(x)\,dx \leqq u_1 + u_2 + \cdots + u_{M-1} \qquad (I)$$

If $f(x)$ is strictly decreasing, the equality signs in ($I$) can be omitted.

If $\lim_{M\to\infty}\int_1^M f(x)\,dx$ exists and is equal to $S$, we see from the left-hand inequality in ($I$) that $u_2 + u_3 + \cdots + u_M$ is monotonic increasing and bounded above by $S$, so that $\Sigma u_n$ converges.

If $\lim_{M\to\infty}\int_1^M f(x)\,dx$ is unbounded, we see from the right-hand inequality in ($I$) that $\Sigma u_n$ diverges.

Thus the proof is complete.

**2.4 Alternating series test**:

An alternating series is one whose successive terms are alternately positive and negative. An alternating series $\sum u_n$ converges if the following two conditions are satisfied.

(*a*) $|u_{n+1}| \leqq |u_n|$ for $n \geqq N$ (Since a fixed number of terms does not affect the convergence or divergence of a series, $N$ may be any positive integer. Frequently it is chosen to be 1.)

(*b*) $\lim_{n\to\infty} u_n = 0 \left(\text{or } \lim_{n\to\infty} |u_n| = 0\right)$

PROOF: Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be an alternating series (here $a_n>0$ for all $n$) satisfying the above conditions (a) and (b).

The sum of the series to $2M$ terms is

$$\begin{aligned} S_{2M} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2M-1} - a_{2M}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2M-2} - a_{2M-1}) - a_{2M} \end{aligned}$$

Since the quantities in parentheses are non-negative, we have

$$S_{2M} \geqq 0, \qquad S_2 \leqq S_4 \leqq S_6 \leqq S_8 \leqq \cdots \leqq S_{2M} \leqq a_1$$

Therefore, $\{S_{2M}\}$ is a bounded monotonic increasing sequence and thus has limit $S$.

Also, $S_{2M+1} = S_{2M} + a_{2M+1}$. Since $\lim_{M\to\infty} S_{2M} = S$ and $\lim_{M\to\infty} a_{2M+1} = 0$ (for, by hypothesis, $\lim_{n\to\infty} a_n = 0$), it follows that $\lim_{M\to\infty} S_{2M+1} = \lim_{M\to\infty} S_{2M} + \lim_{M\to\infty} a_{2M+1} = S + 0 = S$.

Thus, the partial sums of the series approach the limit $S$ and the series converges.

**EXAMPLE.** For the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, we have $u_n = \frac{(-1)^{n-1}}{n}$, $|u_n| = \frac{1}{n}$, $|u_{n+1}| = \frac{1}{n+1}$. Then for $n \geqq 1$, $|u_{n+1}| \leqq |u_n|$. Also $\lim_{n\to\infty} |u_n| = 0$. Hence, the series converges.

**2.5 Absolute and conditional convergence.**

**Definition:** The series $\sum_{n=1}^{\infty} u_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |u_n|$ converges. If $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} |u_n|$ diverges, then $\sum_{n=1}^{\infty} u_n$ is called *conditionally convergent.*

**Lemma:** The absolutely convergent series is convergent.

PROOF:

Given that $\Sigma|u_n|$ converges, we must show that $\Sigma u_n$ converges.

Let $S_M = u_1 + u_2 + \cdots + u_M$ and $T_M = |u_1| + |u_2| + \cdots + |u_M|$. Then

$$\begin{aligned} S_M + T_M &= (u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_M + |u_M|) \\ &\leqq 2|u_1| + 2|u_2| + \cdots + 2|u_M| \end{aligned}$$

Since $\Sigma|u_n|$ converges and since $u_n + |u_n| \geqq 0$, for $n = 1, 2, 3, \ldots$, it follows that $S_M + T_M$ is a bounded monotonic increasing sequence, and so $\lim_{M\to\infty} (S_M + T_M)$ exists.

Also, since $\lim_{M\to\infty} T_M$ exists (since the series is absolutely convergent by hypothesis),

$$\lim_{M\to\infty} S_M = \lim_{M\to\infty} (S_M + T_M - T_M) = \lim_{M\to\infty} (S_M + T_M) - \lim_{M\to\infty} T_M$$

must also exist and the result is proved.

**EXAMPLE 1.** $\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\cdots$ is absolutely convergent and thus convergent, since the series of absolute values $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ converges.

**EXAMPLE 2.** $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges, but $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges. Thus, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent.

Any of the tests used for series with non-negative terms can be used to test for absolute convergence. Also, tests that compare successive terms are common. Tests 6, 8, and 9 are of this type.

**2.6 Ratio (D'Alembert) Test:**

Let $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L$. Then the series $\Sigma u_n$

(*a*) converges (absolutely) if $L < 1$

(*b*) diverges if $L > 1$.

If $L = 1$ the test fails.

PROOF: a) Since $L<1$, we can take an $\varepsilon > 0$ such that $0<L+\varepsilon<1$. Then there exists an $n_0$ such that $\left|\frac{u_{n+1}}{u_n}\right| < L+\varepsilon$ for all $n\geq N$. Therefore, it follows that $|u_{n+1}|<|u_n|(L+\varepsilon)$ for all $n\geq N$. Hence, $|u_n|<|u_{n-1}|(L+\varepsilon)< |u_{n-2}|(L+\varepsilon)^2<\ldots<|u_N|(L+\varepsilon)^{n-N}$ for all $n>N$.

Since $\sum_{n=1}^{\infty}|u_N|(L+\varepsilon)^n$ is convergent, it follows that $\sum_{n=1}^{\infty}|u_n|$ is convergent by comparison test. It means that $\sum_{n=1}^{\infty}u_n$ is absolutely convergent.

b) If $L>1$ then $|u_{n+1}|>|u_n|$ for sufficiently large n. Therefore, $\{u_n\}$ does not tend to 0 when n tends to infinity. This follows that $\sum_{n=1}^{\infty}u_n$ diverges.

If $L=1$, we take $\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Both of them satisfy $L=1$, but the former diverges and the latter converges.

**EXAMPLE:** $\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n!}$ converges absolutely, since $\lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}2^{n+1}}{(n+1)!}\right|}{\left|\frac{(-1)^n 2^n}{n!}\right|} = \lim_{n\to\infty}\frac{2}{n+1} = 0<1.$

The following test can be proved by the same manner.

**2.7 The *n*th root (Cauchy) Test:**

Let $\lim_{n\to\infty}\sqrt[n]{|u_n|} = L$. Then the series $\Sigma u_n$

(*a*) converges (absolutely) if $L < 1$

(*b*) diverges if $L > 1$.

If $L = 1$ the test fails.

8. **Raabe's test**. Let $\lim_{n\to\infty} n\left(1 - \left|\frac{u_n + 1}{u_n}\right|\right) = L$. Then the series $\Sigma u_n$

   (*a*) converges (absolutely) if $L > 1$

   (*b*) diverges or converges conditionally if $L < 1$.

   If $L = 1$ the test fails.

   This test is often used when the ratio tests fails.

9. **Gauss' test**. If $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^2}$, where $|c_n| < P$ for all $n > N$, then the series $\Sigma u_n$

   (*a*) converges (absolutely) if $L > 1$

   (*b*) diverges or converges conditionally if $L \leqq 1$.

   This test is often used when Raabe's test fails.

## 3. Theorem on Absolutely Convergent Series

**Theorem 4.** (Rearrangement of Terms) The terms of an absolutely convergent series can be rearranged in any order, and all such rearranged series will converge to the same sum. However, if the terms of a conditionally convergent series are suitably rearranged, the resulting series may diverge or converge to any desired sum.

**Theorem 5.** (Sums, Differences, and Products) The sum, difference, and product of two absolutely convergent series is absolutely convergent. The operations can be performed as for finite series.

# CHAPTER 2: INFINITE SEQUENCES AND SERIES OF FUNCTIONS

We open this chapter with the thought that functions could be expressed in series form. Such representation is illustrated by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

where

$$\sin x = \lim_{n\to\infty} S_n, \quad \text{with} \quad S_1 = x, S_2 = x - \frac{x^3}{3!}, \ldots S_n = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}.$$

Observe that until this section the sequences and series depended on one element, n. Now there is variation with respect to x as well. This complexity requires the introduction of a new concept called uniform convergence, which, in turn, is fundamental in exploring the continuity, differentiation, and integrability of series.

## 1. Basic Concepts of Sequences and Series of Functions

### 1.1 Definitions:

Let $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ be a sequence of functions defined in $[a, b]$. The sequence is said to converge to $F(x)$, or to have the limit $F(x)$ in $[a, b]$, if for each $\epsilon > 0$ and each $x$ in $[a, b]$ we can find $N > 0$ such that $|u_n(x) - F(x)| < \epsilon$ for all $n > N$. In such case we write $\lim_{n\to\infty} u_n(x) = F(x)$. The number $N$ may depend on $x$ as well as $\epsilon$. If it depends *only* on $\epsilon$ and not on $x$, the sequence is said to converge to $F(x)$ *uniformly* in $[a, b]$ or to be *uniformly convergent* in $[a, b]$.

The infinite series of functions

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots \tag{3}$$

is said to be convergent in [a, b] if the sequence of partial sums $\{S_n(x)\}$, $n$= 1,2,3,..., where $S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$, is convergent in [a, b]. In such case we write $\lim_{n\to\infty} S_n = S(x)$ and call S(x) the sum of the series.

It follows that $\Sigma u_n(x)$ converges to $S(x)$ in $[a, b]$ if for each $\epsilon > 0$ and each $x$ in $[a, b]$ we can find $N > 0$ such that $|S_n(x) - S(x)| < \epsilon$ for all $n > N$. If $N$ depends *only* on $\epsilon$ and not on $x$, the series is called *uniformly convergent* in $[a, b]$.

Since $S(x) - S_n(x) = R_n(x)$, the remainder after $n$ terms, we can equivalently say that $\Sigma u_n(x)$ is uniformly convergent in $[a, b]$ if for each $\epsilon > 0$ we can find $N$ depending on $\epsilon$ but not on $x$ such that $|R_n(x)| < \epsilon$ for all $n > N$ and all $x$ in $[a, b]$.

These definitions can be modified to include other intervals besides [a, b], such as (a, b), and so on.

The **domain of convergence** (absolute or uniform) of a series is the set of values of x for which the series of functions converges (absolutely or uniformly).

**EXAMPLE 1.** Suppose $u_n(x) = x^n/n$ and $-1/2 \le x \le 1$. Now, think of the constant function $F(x) = 0$ on this interval. For any $\varepsilon > 0$ and any x in the interval, there is N such that for all n > N we have $|u_n(x) - F(x)| < \varepsilon$, i.e., $|x^n/n| < \varepsilon$. Since the limit does not depend on x, the sequence is uniformly convergent.

**EXAMPLE 2.** If $u_n = x^n$ and $0 \leqq x \leqq 1$, the sequence is not uniformly convergent because (think of the function $F(x) = 0$, $0 \leqq x < 1$, $F(1) = 1$)

$$|x^n - 0| < \epsilon \text{ when } x^n < \epsilon,$$

thus

$$n \ln x < \ln \epsilon.$$

On the interval $0 \leqq x < 1$, and for $0 < \epsilon < 1$, both members of the inequality are negative, therefore, $n > \frac{\ln \epsilon}{\ln x}$. Since $\frac{\ln \epsilon}{\ln x} = \frac{\ln 1 - \ln \epsilon}{\ln 1 - nn\, x} = \frac{\ln(/\epsilon)}{\ln(1/x)}$, it follows that we must choose $N$ such that

$$n > N > \frac{\ln 1/\epsilon}{\ln 1/x}$$

From this expression we see that $\epsilon \to 0$ then $\ln \frac{1}{\epsilon} \to \infty$ and also as $x \to 1$ from the left $\ln \frac{1}{x} \to 0$ from the right; thus, in either case, $N$ must increase without bound. This dependency on both $\epsilon$ and $x$ demonstrations that the sequence is not uniformly convergent. For a pictorial view of this example, see Fig. 11-1.

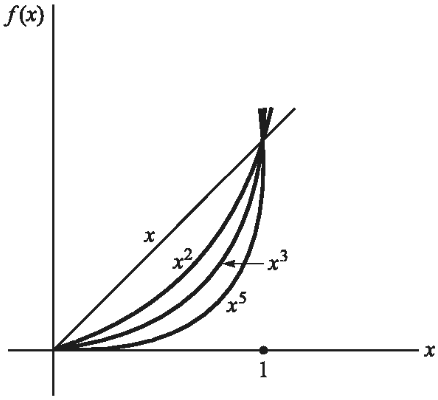



**Fig. 11-1**

### 1.2 Special tests for uniform convergence of series

**1. Weierstrass M** test. If sequence of positive constants $M_1, M_2, M_3, \ldots$, can be found such that in some interval

(a) $|u_n(x)| \leq M_n$, n= 1,2,3,... for all x in this interval

(b) $\sum_{n=1}^{\infty} M_n$ converges

then $\sum_{n=1}^{\infty} u_n(x)$ is uniformly and absolutely convergent in the interval.

**EXAMPLE.** $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly and absolutely convergent in $[0, 2\pi]$ since $\left|\frac{\cos nx}{n^2}\right| \leqq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.

This test supplies a sufficient but not a necessary condition for uniform convergence, i.e., a series may be uniformly convergent even when the test cannot be made to apply.

One may be led because of this test to believe that uniformly convergent series must be absolutely convergent, and conversely. However, the two properties are independent, i.e., a series can be uniformly convergent without being absolutely convergent, and conversely.

2. **Dirichlet's test.** Suppose that

(*a*) the sequence $\{a_n\}$ is a monotonic decreasing sequence of positive constants having limit zero,

(*b*) there exists a constant $P$ such that for $a \leqq x \leqq b$

$$|u_1(x) + u_2(x) + \cdots + u_n(x)| < P \qquad \text{for all } n > N.$$

Then the series

$$a_1 u_1(x) + a_2 u_2(x) + \cdots = \sum_{n=1}^{\infty} a_n u_n(x)$$

is uniformly convergent in $a \leqq x \leqq b$.

## 2. Theorems on uniformly convergent series

If an infinite series of functions is uniformly convergent, it has many of the properties possessed by sums of finite series of functions, as indicated in the following theorems.

**Theorem 6.** If $\{u_n\{x\}\}$, n= 1,2, 3,... are continuous in [a, b] and if $\sum u_n(x)$ converges uniformly to the sum S(x) in [a, b], then S(x) is continuous in [a, b].

Briefly, this states that a uniformly convergent series of continuous functions is a continuous function. This result is often used to demonstrate that a given series is not uniformly convergent by showing that the sum function S(x) is discontinuous at some point.
In particular if $x_0$ is in [a, b], then the theorem states that

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0)$$

***Theorem 7.*** If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$, are continuous in $[a, b]$ and if $\Sigma u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \tag{4}$$

or

$$\int_a^b \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \tag{5}$$

Briefly, a uniformly convergent series of continuous functions can be integrated term by term.

Considering the differentiability we have the following theorem.

***Theorem 8.*** If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$, are continuous and have continuous derivatives in $[a, b]$ and if $\Sigma u_n(x)$ converges to $S(x)$ while $\Sigma u_n'(x)$ is uniformly convergent in $[a, b]$, then in $[a, b]$

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x) \tag{6}$$

or

$$\frac{d}{dx}\left\{\sum_{n=1}^{\infty} u_n(x)\right\} = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x) \tag{7}$$

## 3. Power Series

**3.1 Definition:**

A series having the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

where $a_0$, $a_1$, $a_2$,…., are constants, is called a power series in $x$. It is often convenient to abbreviate the above series as $\sum a_n x^n$ .

**3.2. Abel's theorem**

If the power series $\sum a_n x^n$ converges at the point $x_0 \neq 0$, then it converges at any point $x$ satisfying $|x|<| x_0|$. Moreover, if it diverges at the point $x_1$, then it diverges at any point $x$ satisfying $|x|>| x_1|$.

PROOF. We prove the first assertion, and the second assertion easily follows from the first one. Let estimate $| a_n x^n | \leq |a_n x_0| \left|\frac{x}{x_0}\right|^n$ .

Since, the series $\sum a_n x_0^n$ converges, we have that $\lim_{n\to\infty} a_n x_0=0$. Therefore, there exists M>0, such that $|a_n x_0| \leq M$ for all $n$. We thus obtain that

$$| a_n x^n | \leq M \left|\frac{x}{x_0}\right|^n \text{ for all } n.$$

Since $|x|<| x_0|$, the assertion now follows from the comparison test.

**General remarks:**

In general, a power series converges for |x| < R and diverges for |x| > R, where the constant R is called the radius of convergence of the series. For |x| = R, the series may or may not converge.

The interval |x| < R or -R < x < R, with possible inclusion of endpoints, is called the ***interval of convergence*** of the series. Although the ratio test is often successful in obtaining this interval, it may fail and in such cases, other tests may be used.

The two special cases R = 0 and R = ∞ can arise. In the first case the series converges only for x = 0; in the second case it converges for all x, sometimes written —∞ < x < ∞.

When we speak of a convergent power series, we shall assume, unless otherwise indicated, that R > 0.

### 3.3 More theorems on power series

**Theorem 9.** A power series converges uniformly and absolutely in any interval which lies entirely within its interval of convergence.

**Theorem 10.** A power series can be differentiated or integrated term by term over any interval lying entirely within the interval of convergence. Also, the sum of a convergent power series is continuous in any interval lying entirely within its interval of convergence.

**Theorem 11.** When a power series converges up to and including an endpoint of its interval of convergence, the interval of uniform convergence also extends so far as to include this endpoint.

***Theorem 12.*** *Abel's limit theorem.* If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, which may be an interior point or an endpoint of the interval of convergence, then

$$\lim_{x \to x_0} \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ \lim_{x \to x_0} a_n x^n \right\} = \sum_{n=0}^{\infty} a_n x_0^n \qquad (10)$$

If $x_0$ is an end point, we must use $x \to x_0+$ or $x \to x_0-$ in (10) according as $x_0$ is a left- or right-hand end point.

### 3.4 Operations with power series

In the following theorems we assume that all power series are convergent in some interval.

**Theorem 13.** Two power series can be added or subtracted term by term for each value of x common to their intervals of convergence.

***Theorem 14.*** Two power series, for example, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, can be multiplied to obtain $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 \qquad (11)$$

the result being valid for each $x$ within the common interval of convergence.

***Theorem 15.*** If the power series $\sum_{n=0}^{\infty} a_n x^n$ is divided by the power series $\Sigma b_n x^n$ where $b_0 \neq 0$, the quotient can be written as a power series which converges for sufficiently small values of $x$.

***Theorem 16.*** If $y = \sum_{n=0}^{\infty} a_n x^n$, then by substituting $x = \sum_{n=0}^{\infty} b_n y^n$, we can obtain the coefficients $b_n$ in terms of $a_n$. This process is often called *reversion of series.*

### 3.5 Expansion of Functions in Power Series

This section gets at the heart of the use of infinite series in analysis. Functions are represented through them. Certain forms bear the names of mathematicians of the eighteenth and early nineteenth century who did so much to develop these ideas.

A simple way (and one often used to gain information in mathematics) to explore series representation of functions is to assume such a representation exists and then discover the details. Of course, whatever is found must be confirmed in a rigorous manner. Therefore, assume

$$f(x) = A_0 + A_1(x - c) + A_2(x - c)^2 + \ldots + A_n(x - c)^n + \ldots$$

Notice that the coefficients $A_n$ can be identified with derivatives of f(x). In particular $A_0=f(c)$, $A_1 = f'(c)$, $A_2= f''(c)/2!,\ldots,A_n=f^{(n)}(c)/n!,\ldots$ This suggests that a series representation of f(x) is

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2!}f''(c)(x - c) + \ldots + \frac{1}{n!}f^{(n)}(c)(x-c) + \ldots$$

A first step in formalizing series representation of a function, f(x), for which the first n derivatives exist, is accomplished by introducing Taylor polynomials of the function.

$P_0(x) = f(c)$; $P_1(x) = f(c) + f'(c)(x - c)$; $P_2(x) = f(c) + f'(c)(x - c) + \frac{1}{2!}f''(c)(x - c)^2$; …

$$P_n(x) = f(c) + f'(c)(x - c) + \bullet\bullet\bullet + \frac{1}{n!}f^{(n)}(c)x-c)^n \qquad (12)$$

**TAYLOR'S THEOREM**

Let $f$ and its derivatives $f', f'', \ldots, f^{(n)}$ exist and be continuous in a closed interval $a \le x \le b$ and suppose that $f^{(n+1)}$ exists in the open interval $a < x < b$. Then for $c$ in $[a, b]$,

$$f(x) = P_n(x) + R_n(x),$$

where the remainder $R_n(x)$ may be represented in any of the three following ways.

For each $n$ there exists $\xi$ such that

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1} \qquad \text{(Lagrange form)} \qquad (13)$$

($\xi$ is between $c$ and $x$.)

(The theorem with this remainder is a mean value theorem. Also, it is called Taylor's formula.)

For each $n$ there exists $\xi$ such that

$$R_n(x) = \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n(x-c) \qquad \text{(Cauchy form)} \qquad (14)$$

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt \qquad \text{(Integral form)} \qquad (15)$$

If all the derivatives of $f$ exist, then the infinite series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \qquad (16)$$

is called a Taylor series of the function *f*, although when c = 0, it can also be referred to as a MacLaurin series or expansion.

The Taylor series of a function may be convergent or divergent (except at the point *c*) on [a, b]. In case it converges on [a, b], the sum may or may not equal f(x). The following theorem gives a sufficient condition for the Taylor (or MacLaurin) series (16) to be convergent to f(x).

**THEOREM.** Let the function *f* have the derivatives of all orders on (-R,R) (with R>0). If there is an **M**>0 such that

$$|f^{(n)}(x)| \leq \mathbf{M} \text{ for all } x \in (-R,R) \text{ and all } n,$$

then the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is convergent to *f*(x) on (-R,R). In other words:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \text{ for all } x \in (-R,R).$$

PROOF. This is direct consequence of the Taylor's formula with Lagrange's Remainder.

**EXAMPLE.** The value of $\sin x$ may be determined geometrically for $0, \frac{\pi}{6}$, and an infinite number of other arguments. To obtain values for other real number arguments, a Taylor series may be expanded about any of these points. For example, let $c = 0$ and evaluate several derivatives there, i.e., $f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0$, $f'''(0) = -\cos 0 = -1$, $f^{1v}(0) = \sin 0 = 0$, $f^{v}(0) = \cos 0 = 1$.

Thus, the MacLaurin expansion to five terms is

$$\sin x = 0 + x - 0 - \frac{1}{3!}x^3 + 0 - \frac{1}{51}x^5 + \cdots$$

Since the fourth term is 0 the Taylor polynomials $P_3$ and $P_4$ are equal, i.e.,

$$P_3(x) = P_4(x) = x - \frac{x^3}{3!}$$

and the Lagrange remainder is

$$R_4(x) = \frac{1}{5!}\cos \xi \, x^5$$

Suppose an approximation of the value of sin .3 is required. Then

$$P_4(.3) = .3 - \frac{1}{6}(.3)^3 \approx .2945.$$

The accuracy of this approximation can be determined from examination of the remainder. In particular, (remember $|\cos \xi| \leq 1$)

$$|R_4| = \left|\frac{1}{5!}\cos \xi (.3)^5\right| \leq \frac{1}{120}\frac{243}{10^5} < .000021$$

Thus, the approximation $P_4(.3)$ for sin .3 is correct to four decimal places.

Additional insight to the process of approximation of functional values results by constructing a graph of $P_4(x)$ and comparing it to $y = \sin x$. (See Fig. 11-2.)

$$P_4(x) = x - \frac{x^3}{6}$$

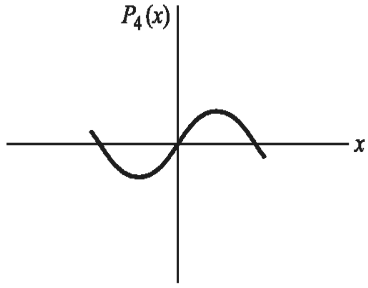


Fig. 11-2

The roots of the equation are $0, \pm\sqrt{6}$. Examination of the first and second derivatives reveals a relative maximum at $x = \sqrt{2}$ and a relative minimum at $x = -\sqrt{2}$. The graph is a local approximation of the sin curve. The reader can show that $P_6(x)$ produces an even better approximation.

(For an example of series approximation of an integral see the example below.)

### 3.6 SOME IMPORTANT POWER SERIES

The following series, convergent to the given function in the indicated intervals, are frequently employed in practice:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots ; \quad -\infty < x < \infty$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots ; \quad -\infty < x < \infty$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots ; \quad -\infty < x < \infty$
4. $\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n-1}\frac{x^n}{n} + \cdots ; \quad -1 < x \leqq 1$
5. $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots ; \quad -1 < x < 1$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots ; \quad -1 \leqq x \leqq 1$
7. $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}x^n + \cdots ;$

## 4. Fourier Series

Mathematicians of the eighteenth century, including Daniel Bernoulli and Leonard Euler, expressed the problem of the vibratory motion of a stretched string through partial differential equations that had no solutions in terms of "elementary functions." Their resolution of this difficulty was to introduce infinite series of sine and cosine functions that satisfied the equations. In the early nineteenth century, Joseph Fourier, while studying the problem of heat flow, developed a cohesive theory of such series.

Consequently, they were named after him. Fourier series are investigated in this section. As you explore the ideas, notice the similarities and differences with the infinite series.

**4.1 Periodic functions:** A function f(x) is said to have a period T or to be periodic with period T if for all x, f{x + T) = f(x), where T is a positive constant. The least value of T > 0 is called the least period or simply the period of f(x).

EXAMPLE 1. The function sinx has periods 2π, 4π, 6π,..., since sin(x + 2π), sin(x + 4π), sin (x +6π),... all equal sinx. However, 2π is the least period or the period of sinx.

EXAMPLE 2. The period of *sinn*πx or *cosn*πx, where *n* is a positive integer, is 2π/n.

EXAMPLE 3. The period of tanx is π.

EXAMPLE 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figures 13-1 (a), (b), and (c) below.

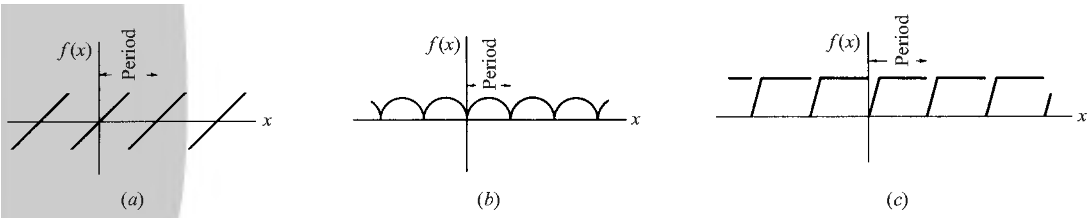



Fig. 13-1

**4.2 Definition of Fourier Series**

Let $f(x)$ be defined in the interval $(-L, L)$ and outside of this interval by $f(x+2L)=f(x)$, i.e., $f(x)$ is $2L$-periodic. It is through this avenue that a new function on an infinite set of real numbers is created from the image on $(-L, L)$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right) \qquad (1)$$

where the *Fourier coefficients* $a_n$ and $b_n$ are

$$\begin{cases} a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n=0,1,2,\ldots \qquad (2)$$

**4.3 Orthogonality Conditions for the Sine and Cosine Functions**

Notice that the Fourier coefficients are integrals. These are obtained by starting with the series (1), and employing the following properties called orthogonality conditions:

(a) $\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$ if $m \neq n$ and $L$ if $m = n$

(b) $\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$ if $m \neq n$ and $L$ if $m = n$ (3)

(c) $\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$. Where $m$ and $n$ can assume any positive integer values.

**EXAMPLE 1.** To determine the Fourier coefficient $a_0$, integrate both sides of the Fourier series (1), i.e.,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L}\frac{a_0}{2}\,dx + \int_{-L}^{L}\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right\}dx$$

Now $\int_{-L}^{L}\frac{a_0}{2}\,dx = a_0 L$, $\int_{-l}^{L}\sin\frac{n\pi x}{L}\,dx = 0$, $\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = 0$, therefore, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

**EXAMPLE 2.** To determine $a_1$, multiply both sides of (1) by $\cos\frac{\pi x}{L}$ and then integrate. Using the orthogonality conditions $(3)_a$ and $(3)_c$, we obtain $a_1 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi x}{L}\,dx$. Now see Problem 13.4.

If $L = \pi$, the series (1) and the coefficients (2) or (3) are particularly simple. The function in this case has the period $2\pi$.

**DIRICHLET CONDITIONS**

Suppose that

(1) $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$

(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (1) with Fourier coefficients converges to

(a) $f(x)$ if $x$ is a point of continuity

(b) $\frac{f(x+0)+f(x-0)}{2}$ if $x$ is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right- and left-hand limits of $f(x)$ at $x$ and represent $\lim_{\epsilon\to 0+} f(x+\epsilon)$ and $\lim_{\epsilon\to 0+} f(x-\epsilon)$, respectively. For a proof see Problems 13.18 through 13.23.

The conditions (1), (2), and (3) imposed on $f(x)$ are *sufficient* but not necessary, and are generally satisfied in practice. There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* ensure convergence of a Fourier series.

## 4.4 Odd and Even Functions

A function f(x) is called odd if f(-x) =-f(x). Thus, $x^3 + x^5 - 3x^3 + 2x$, sin x, tan 3x are odd functions.

A function f(x) is called even if f(-x)=f(x). Thus, $x^2$, $2x^4 - 4x^2 + 5$, cos x, $e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Figures 13-1 (a) and 13-1 (b) are odd and even respectively, but that of Fig. 13-l(c) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant which we shall consider a cosine term) can be present.

**4.5 Half Range Fourier Sine or Cosine Series.**

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0, L) [which is half of the interval (-L, L), thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other
half of the interval, namely, (-L, 0). In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx & \text{for } \textit{half range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx & \text{for } \textit{half range cosine series} \end{cases} \tag{4}$$

**4.6 Parseval's Identity**

If $a_n$ and $b_n$ are the Fourier coefficients corresponding to *f(x)* and if *f(x)* satisfies the Dirichlet conditions. Then

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{5}$$

**4.7 Differentiation and Integration of Fourier Series.**

Differentiation and integration of Fourier series can be justified by using the previous theorems, which hold for series in general. It must be emphasized, however, that those theorems provide sufficient conditions and are not necessary. The following theorem for integration is especially useful.

***Theorem.*** The Fourier series corresponding to $f(x)$ may be integrated term by term from $a$ to $x$, and the resulting series will converge uniformly to $\int_a^x f(x)\,dx$ provided that $f(x)$ is piecewise continuous in $-L \leqq x \leqq L$ and both $a$ and $x$ are in this interval.

**4.8 Complex Notation for Fourier Series**

Using Euler's identities:

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta \tag{6}$$

where $i = \sqrt{-1}$ , the Fourier series for $f(x)$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{7}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx \tag{8}$$

In writing the equality (7) we are supposing that the Dirichlet conditions are satisfied and further that $f(x)$ is continuous at $x$. If $f(x)$ is discontinuous at $x$, the left side of (7) should be replaced by $\frac{(f(x+0)+f(x-0))}{2}$.

# Problems

**CONVERGENCE AND DIVERGENCE OF SERIES OF CONSTANTS**

1. (*a*) Prove that the series $\frac{1}{3\cdot 7}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 15}+\cdots=\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n+3)}$ converges and (*b*) find its sum.
*Ans.* (*b*) 1/12

2. Prove that the convergence or divergence of a series is not affected by (*a*) multiplying each term by the same non-zero constant, (*b*) removing (or adding) a finite number of terms.

3. If $\Sigma u_n$ and $\Sigma v_n$ converge to $A$ and $B$, respectively, prove that $\Sigma(u_n+v_n)$ converges to $A+B$.

4. Prove that the series $\frac{3}{2}+(\frac{3}{2})^2+(\frac{3}{2})^3+\cdots=\Sigma(\frac{3}{2})^n$ diverges.

5. Find the fallacy: Let $S=1-1+1-1+1-1+\cdots$. Then $S=1-(1-1)-(1-1)-\cdots=1$ and $S=(1-1)+(1-1)+(1-1)+\cdots=0$. Hence, $1=0$.

**COMPARISON TEST AND QUOTIENT TEST**

6. Test for convergence:

(*a*) $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$, (*b*) $\sum_{n=1}^{\infty}\frac{n}{4n^2-3}$, (*c*) $\sum_{n=1}^{\infty}\frac{n+2}{(n+1)\sqrt{n+3}}$, (*d*) $\sum_{n=1}^{\infty}\frac{3^n}{n\cdot 5^n}$, (*e*) $\sum_{n=1}^{\infty}\frac{1}{5n-3}$,

(*f*) $\sum_{n=1}^{\infty}\frac{2n-1}{(3n+2)n^{4/3}}$.

*Ans.* (*a*) conv., (*b*) div., (*c*) div., (*d*) conv., (*e*) div., (*f*) conv.

7. Investigate the convergence of (*a*) $\sum_{n=1}^{\infty}\frac{4n^2+5n-2}{n(n^2+1)^{3/2}}$, (*b*) $\sum_{n=1}^{\infty}\sqrt{\frac{n-\ln n}{n^2+10n^3}}$. *Ans.* (*a*) conv., (*b*) div.

8. Establish the comparison test for divergence.

9. Use the comparison test to prove that

(a) diverges if $p \leqq 1$ (b) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n}$ diverges, (c) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges

10. Establish the results (b) and (c) of the quotient test

Test for convergence:

(a) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$, (b) $\sum_{n=1}^{\infty} \sqrt{n \tan^{-1}(1/n^3)}$, (c) $\sum_{n=1}^{\infty} \frac{3 + \sin n}{n(1 + e^{-n})}$, (d) $\sum_{n=1}^{\infty} n \sin^2(1/n)$.

*Ans.* (a) conv., (b) div., (c) div., (d) div.

11. If $\Sigma u_n$ converges, where $u_n \geqq 0$ for $n > N$, and if $\lim_{n \to \infty} nu_n$ exists, prove that $\lim_{n \to \infty} nu_n = 0$.

(a) Test for convergence $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$. (b) Does your answer to (a) contradict the statement about the p series that $\Sigma 1/n^p$ converges for $p > 1$?
*Ans.* (a) div.

## INTEGRAL TEST

12. Test for convergence: (a) $\sum_{n=1}^{\infty} \frac{n^2}{2n^3 - 1}$, (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$, (c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$, (d) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ (e) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$,
(f) $\sum_{n=10}^{\infty} \frac{2^{\ln(\ln n)}}{n \ln n}$. (g) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

*Ans.* (a) div., (b) conv., (c) conv., (d) conv., (e) div., (f) div. (g) conv if $p > 1$, div if $p \leqq 1$.

13. Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, where $p$ is a constant, (a) converges if $p > 1$ and (b) diverges if $p \leqq 1$.

Prove that $\frac{9}{8} < \sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{5}{4}$.

14. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{e^{\tan^{-1} n}}{n^2 + 1}$.
*Ans.* conv.

(a) Prove that $\frac{2}{3}n^{3/2} + \frac{1}{3} \leqq \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leqq \frac{2}{3}n^{3/2} + n^{1/2} - \frac{2}{3}$.
(b) Use (a) to estimate the value of $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}$, giving the maximum error.
(c) Show how the accuracy in (b) can be improved by estimating, for example, $\sqrt{10} + \sqrt{11} + \cdots + \sqrt{100}$ and adding on the value of $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{9}$ computed to some desired degree of accuracy.
*Ans.* (b) $671.5 \pm 4.5$

## ALTERNATING SERIES

15. Test for convergence: (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$, (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 2n + 2}$, (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n - 1}$,
(d) $\sum_{n=1}^{\infty} (-1)^n \sin^{-1} \frac{1}{n}$, (e) $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$.

*Ans.* (a) conv., (b) conv., (c) div., (d) conv., (e) div.

16. (*a*) What is the largest absolute error made in approximating the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(n+1)}$ by the sum of the first 5 terms?
*Ans.* 1/192
(*b*) What is the least number of terms which must be taken in order that 3 decimal place accuracy will result?
*Ans.* 8 terms

17. (*a*) Prove that $S = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \frac{4}{3}\left(\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots\right)$.
(*b*) How many terms of the series on the right are needed in order to calculate $S$ to six decimal place accuracy?
*Ans.* (*b*) at least 100 terms

## ABSOLUTE AND CONDITIONAL CONVERGENCE

18. Test for absolute or conditional convergence:

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ (*c*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ (*e*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sin \frac{1}{\sqrt{n}}$

(*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$ (*d*) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{(n^2+1)^{4/3}}$ (*f*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3}{2^n - 1}$

*Ans.* (*a*) abs. conv., (*b*) cond. conv., (*c*) cond. conv., (*d*) div., (*e*) abs. conv., (*f*) abs. conv.

Prove that $\sum_{n=1}^{\infty} \frac{\cos n\pi a}{x^2+n^2}$ converges absolutely for all real $x$ and $a$.

If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $S$, prove that the rearranged series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}S$. Explain.
[Hint: Take 1/2 of the first series and write it as $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$; then add term by term to the first series. Note that $S = \ln 2$, as shown in Problem 11.100.]

19. Prove that the terms of an absolutely convergent series can always be rearranged without altering the sum.

## RATIO TEST

20. Test for convergence:

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)e^n}$, (*b*) $\sum_{n=1}^{\infty} \frac{10^{2n}}{(2n-1)!}$, (*c*) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$, (*d*) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{3n}}{3^{2n}}$, (*e*) $\sum_{n=1}^{\infty} \frac{(\sqrt{5}-1)^n}{n^2+1}$.

*Ans.* (*a*) conv. (abs.), (*b*) conv., (*c*) div., (*d*) conv. (abs.), (*e*) div.

21. Show that the ratio test cannot be used to establish the conditional convergence of a series.

Prove that (*a*) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges and (*b*) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

## MISCELLANEOUS TESTS

22. Establish the validity of the $n$th root test

23. Apply the $n$th root test to work Problems 20 (*a*), (*c*), (*d*), and (*e*).

24. Prove that $\frac{1}{3} + (\frac{2}{3})^2 + (\frac{1}{3})^3 + (\frac{2}{3})^4 + (\frac{1}{3})^5 + (\frac{2}{3})^6 + \cdots$ converges.

25. Test for convergence: (*a*) $\frac{1}{3}+\frac{1\cdot 4}{3\cdot 6}+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}+\cdots$, (*b*) $\frac{2}{9}+\frac{2\cdot 5}{9\cdot 12}+\frac{2\cdot 5\cdot 8}{9\cdot 12\cdot 15}+\cdots$.
*Ans.* (*a*) div., (*b*) conv.

26. If $a, b$, and $d$ are positive numbers and $b > a$, prove that

$$\frac{a}{b}+\frac{a(a+d)}{b(b+d)}+\frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}+\cdots$$

converges if $b-a>d$, and diverges if $b-a \leqq d$.

## SERIES OF FUNCTIONS

27. Find the domain of convergence of the series:

(*a*) $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$, (*b*) $\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{2^n(3n-1)}$, (*c*) $\sum_{n=1}^{\infty}\frac{1}{n(1+x^2)^n}$, (*d*) $\sum_{n=1}^{\infty} n^2\left(\frac{1-x}{1+x}\right)^n$, (*e*) $\sum_{n=1}^{\infty}\frac{e^{nx}}{n^2-n+1}$

*Ans.* (*a*) $-1 \leqq x \leqq 1$, (*b*) $-1 < x \leqq 3$, (*c*) all $x \neq 0$, (*d*) $x > 0$, (*e*) $x \leqq 0$

28. Prove that $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n$ converges for $-1 \leqq x < 1$.

## UNIFORM CONVERGENCE

29. By use of the definition, investigate the uniform convergence of the series

$$\sum_{n=1}^{\infty}\frac{x}{[1+(n-1)x][1+nx]}$$

[Hint: Resolve the $n$th term into partial fractions and show that the $n$th partial sum is $S_n(x) = 1-\frac{1}{1+nx}$.]
*Ans.* Not uniformly convergent in any interval which includes $x=0$; uniformly convergent in any other interval.

30. Investigate by any method the convergence and uniform convergence of the series:

(*a*) $\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$, (*b*) $\sum_{n=1}^{\infty}\frac{\sin^2 nx}{2^n-1}$, (*c*) $\sum_{n=1}^{\infty}\frac{x}{(1+x)^n}$, $x \geqq 0$.

*Ans.* (*a*) conv. for $|x|<3$; unif. conv. for $|x| \leqq r < 3$. (*b*) unif. conv. for all $x$. (*c*) conv. for $x \geqq 0$; not unif. conv. for $x \geqq 0$, but unif. conv. for $x \geqq r > 0$.

31. If $F(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$, prove that:
(*a*) $F(x)$ is continuous for all $x$, (*b*) $\lim_{x\to 0} F(x) = 0$, (*c*) $F'(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ is continous everywhere.

32. Prove that $\int_0^{\pi}\left(\frac{\cos 2x}{1\cdot 3}+\frac{\cos 4x}{3\cdot 5}+\frac{\cos 6x}{5\cdot 7}+\cdots\right)dx = 0$.

33. Prove that $F(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{\sinh n\pi}$ has derivatives of all orders for any real $x$.

34. Examine the sequence $u_n(x) = \frac{1}{1+x^{2n}}$, $n = 1, 2, 3, \ldots$, for uniform convergence.

35. Prove that $\lim_{n\to\infty}\int_0^1\frac{dx}{(1+x/n)^n} = 1-e^{-1}$.

## POWER SERIES

36. (*a*) Prove that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.
(*b*) Prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.
$\left[\text{Hint: Use the fact that } \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \text{ and integrate.}\right]$

37. Prove that $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots, \ -1 \leqq x \leqq 1$.

38. Evaluate (*a*) $\int_0^{1/2} e^{-x^2}\,dx$, (*d*) $\int_0^1 \frac{1-\cos x}{x}\,dx$ to 3 decimal places, justifying all steps.
*Ans.* (*a*) 0.461, (*b*) 0.486

39. Evaluate (*a*) $\sin 40°$, (*b*) $\cos 65°$, (*c*) $\tan 12°$ correct to 3 decimal places.
*Ans.* (*a*) 0.643, (*b*) 0.423, (*c*) 0.213

40. By multiplying the series for $\sin x$ and $\cos x$, verify that $2\sin x\cos x = \sin 2x$.

41. Show that $e^{\cos x} = e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right), \ -\infty < x < \infty$.

42. Obtain the expansions

(*a*) $\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \qquad -1 < x < 1$

(*b*) $\ln(x + \sqrt{x^2+1}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \qquad -1 \leqq x \leqq 1$

43. Let $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that the formal Taylor series about $x = 0$ corresponding to $f(x)$ exists but that it does not converge to the given function for any $x \neq 0$.

44. Prove that

(*a*) $\frac{\ln(1+x)}{1+x} = x - \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 - \cdots \qquad \text{for } -1 < x < 1$

(*b*) $\{\ln(1+x)\}^2 = x^2 - \left(1 + \frac{1}{2}\right)\frac{2x^3}{3} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{2x^4}{4} - \cdots \qquad \text{for } -1 < x \leqq 1$

## MISCELLANEOUS PROBLEMS

45. Prove that the series for $J_p(x)$ converges (*a*) for all $x$, (*b*) absolutely and uniformly in any finite interval.

46. Prove that (*a*) $\frac{d}{dx}\{J_0(x)\} = -J_1(x)$, (*b*) $\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$, (*c*) $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$.

47. Assuming that the result of Problem 11.111(*c*) holds for $p = 0, -1, -2, \ldots$, prove that
(*a*) $J_{-1}(x) = -J_1(x)$, (*b*) $J_{-2}(x) = J_2(x)$, (*c*) $J_{-n}(x) = (-1)^n J_n(x)$, $n = 1, 2, 3, \ldots$.

48. Prove that $e^{1/2x(t-1/t)} = \sum_{p=-\infty}^{\infty} J_p(x)\,t^p$.
[Hint: Write the left side as $e^{xt/2}e^{-x/2t}$, expand and use Problem 11.112.]

**FOURIER SERIES**

49. Graph each of the following functions and find their corresponding Fourier series using properties of even and odd functions wherever applicable.

(a) $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$ Period 4 (b) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(c) $f(x) = 4x, 0 < x < 10,$ Period 10 (d) $f(x) = \begin{cases} 2x & 0 \leqq x < 3 \\ 0 & -3 < x < 0 \end{cases}$ Period 6

*Ans.* (a) $\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{2}$ (b) $2-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^2}\cos\frac{n\pi x}{4}$

(c) $20-\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}$ (d) $\frac{3}{2}+\sum_{n=1}^{\infty}\left\{\frac{6(\cos n\pi - 1)}{n^2\pi^2}\cos\frac{n\pi x}{3}-\frac{6\cos n\pi}{n\pi}\sin\frac{n\pi x}{3}\right\}$

50. In each part of Problem 49, tell where the discontinuities of $f(x)$ are located and to what value the series converges at the discontunities.

*Ans.* (a) $x = 0, \pm 2, \pm 4, \ldots; 0$ (b) no discontinuities (c) $x = 0, \pm 10, \pm 20, \ldots; 20$
(d) $x = \pm 3, \pm 9, \pm 15, \ldots; 3$

51. Expand $f(x) = \begin{cases} 2-x & 0 < x < 4 \\ x-6 & 4 < x < 8 \end{cases}$ in a Fourier series of period 8.

*Ans.* $\frac{16}{\pi^2}\left\{\cos\frac{\pi x}{4}+\frac{1}{3^2}\cos\frac{3\pi x}{4}+\frac{1}{5^2}\cos\frac{5\pi x}{4}+\cdots\right\}$

52. (a) Expand $f(x) = \cos x, 0 < x < \pi$, in a Fourier sine series.
(b) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

*Ans.* (a) $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2nx}{4n^2-1}$ (b) $f(0) = f(\pi) = 0$

53. (a) Expand in a Fourier series $f(x) = \cos x, 0 < x < \pi$ if the period is $\pi$; and (b) compare with the result of Problem 52, explaining the similarities and differences if any.

54. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8-x & 4 < x < 8 \end{cases}$ in a series of (a) sines, (b) cosines.

*Ans.* (a) $\frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8}$ (b) $\frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{2\cos n\pi/2-\cos n\pi - 1}{n^2}\right)\cos\frac{n\pi x}{8}$

55. Prove that for $0 \leqq x \leqq \pi$,

(a) $x(\pi - x) = \frac{\pi^2}{6}-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\cdots\right)$

(b) $x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5}{5^3}+\cdots\right)$

56. Use the preceding problem to show that

(a) $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$, (c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}=\frac{\pi^3}{32}$.

57. Show that $\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}-\cdots=\frac{3\pi^2\sqrt{2}}{16}$.

# CHAPTER 3: BASIC CONCEPT OF DIFFERENTIAL EQUATIONS

In this chapter we provide the readers with some fundamental concepts of differential equations such as solutions and order of differential equations, initial-value problems, standard and differential forms, etc. We first start by considering some examples of differential equations arising from processes in biology, physic, and so on.

## 1. Examples of Differential Equations

### 1.1 Growth and Decay Problems

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that $dN/dt$, the time rate of change of this amount of substance, is proportional to the amount of substance present, this means that $dN/dt = kN$, or

$$dN/dt - kN=0 \qquad (1.1)$$

where $k$ is the constant of proportionality.

We are assuming that $N(t)$ is a differentiable, hence continuous, function of time. For population problems, where $N(t)$ is actually discrete and integer-valued, this assumption is incorrect. Nonetheless, (1.1) still provides a good approximation to the physical laws governing such a system.

### 1.2 Temperature Problems

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let T denote the temperature of the body and let Tm denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is dT/dt, and Newton's law of cooling can be formulated as $dT/dt = -k(T- T_m)$, or as

$$\frac{dT}{dt} + kT = kT_m \qquad (1.2)$$

where k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make dT/dt negative in a cooling process, when T is greater than $T_m$, and positive in a heating process, when T is less than $T_m$.

### 1.3 Falling Body Problems

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction. For the problem at hand, there are two forces acting on the body: the force due to gravity given by the weight m of the body, which equals mg, and the force due to air resistance given by -kv, where $k > 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the upward, or negative, direction (see Figure 1 -1). The net force F on the body is, therefore, F = mg-kv. Using this result and Newton second law of motion (F=mdv/dt), we obtain

$$mg - kv = m\frac{dv}{dt}$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

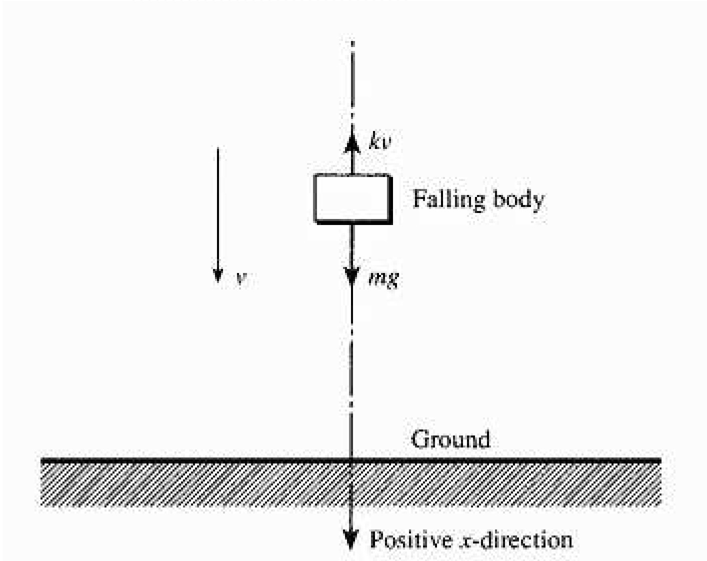



Figure 1.1

**1.4 Electrical Circuits**

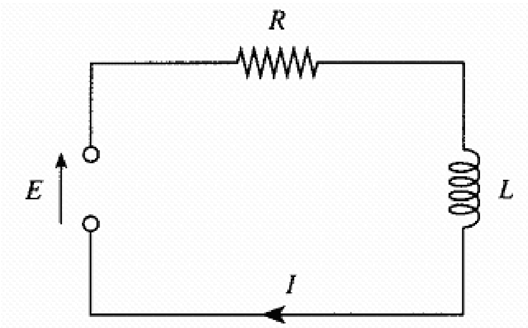



Figure 1-2

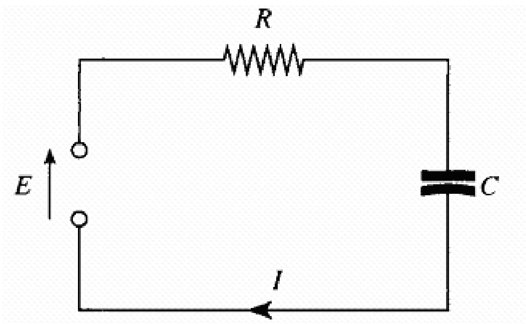



Figure 1-3

The basic equation governing the amount of current / (in amperes) in a simple RL circuit (see Figure 1-2) consisting of a resistance R (in ohms), an inductor L (in henries), and an electromotive force (abbreviated emf) E (in volts) is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

For an RC-circuit consisting of a resistance, a capacitance C (in farads), an emf, and no inductance (Figure 1-3), the equation governing the amount of electrical charge q (in coulombs) on the capacitor is

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$$

The relationship between q and I is q = dI/dt

## 2. Definitions and Related Concepts

**2.1 Definition.** A **differential equation** is an equation involving an unknown function and its derivatives.

The following are differential equations involving the unknown function y.

$$\frac{dy}{dx} = 5x + 3$$

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1$$

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0$$

A differential equation is an **ordinary differential equation** if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is **a partial differential equation.**

The **order** of a differential equation is the order of the highest derivative appearing in the equation.

**2.2 Solution.** A solution of a differential equation in the unknown function y and the independent variable x on the interval *J* is a function y(x) that satisfies the differential equation identically for all x in *J*.

**Example:** The function $y(x) = c_1\sin 2x + c_2\cos 2x$, where $c_1$ and $c_2$ are arbitrary constants, is a solution of $y'' + 4y = 0$ in the interval $(-\infty, \infty)$.

**2.3 Particular and general solutions.**

A particular solution of a differential equation is any one solution. The general solution of a differential equation is the set of all solutions.

**2.4 Initial-Value and Boundary-Value Problems.**

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an initial-value problem. The subsidiary conditions are initial conditions. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a boundary-value problem and the conditions are boundary conditions.

**Example:** The problem $y'' + 2y = x$; $y(\pi) = 1, y'(\pi) = 2$ is an initial value problem, because the two subsidiary conditions are both given at $x = \pi$. The problem $y'' + 2y' = x$; $y(0) = 1$, $y(l) = 1$

is a boundary-value problem, because the two subsidiary conditions are given at $x = 0$ and $x=1$.
A solution to an initial-value or boundary-value problem is a function y(x) that both solves the differential equation and satisfies all given subsidiary conditions.

## 2.5 Standard and Differential Forms

Standard form for a first-order differential equation in the unknown function y(x) is

$$y' = f(x, y) \qquad (2.1)$$

where the derivative y' appears only on the left side of (2.1). Many, but not all, first-order differential equations can be written in standard form by algebraically solving for y' and then setting f(x,y) equal to the right side of the resulting equation. The right side of (2.1) can always be written as a quotient of two other functions -M(x,y) and N(x,y). Then (2.1) becomes $dy/dx = -M(x,y)/N(x, y)$, which is equivalent to the differential form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (2.2)$$

# CHAPTER 4: SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

In this chapter we will consider the solutions of some first-order differential equations. Starting form separable equations we will construct the method to solve more complicated equation such as homogeneous, exact, linear, and Bernoulli equations.

## 1. Separable Equations

**1.1 Definition:** Consider a differential equation in differential form (1.4). If M(x,y) =A(x) (a function only of x) and N(x,y) = B(y) (a function only of y), differential equation is separable, or has its variables separated.

**1.2 General Solution:** The solution to the first-order separable differential equation

$$A(x)dx + B(y)dy = 0 \qquad (1.1)$$

is

$$\int A(x)dx + \int B(y)dy = c \qquad (1.2)$$

where *c* represents an arbitrary constant.

**Example.** Solve the equation: $\frac{dy}{dx} = \frac{x^2+2}{y}$

This equation may be rewritten in the differential form

$$(x^2+2)dx - ydy = 0$$

which is separable with $A(x) = x^2 + 2$ and $B(y) = -y$. Its solution is

$$\int (x^2+2)dx - \int ydy = c$$

or

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c\,.$$

The integrals obtained in Equation (1.2) may be, for all practical purposes, impossible to evaluate. In such case, numerical techniques are used to obtain an approximate solution. Even if the indicated integrations in (1.2) can be performed, it may not be algebraically possible to solve for *y* explicitly in terms of *x*. In that case, the solution is left in implicit form.

**1.3 Solutions to the Initial-Value Problem:**

The solution to the initial-value problem

$$A(x)dx + B(y)dy = 0;\; y(x_0) = y_0 \qquad (1.3)$$

can be obtained, as usual, by first using Equation (1.2) to solve the differential equation and then applying the initial condition directly to evaluate c.

Alternatively, the solution to Equation (1.3) can be obtained from

$$\int_{x_0}^{x} A(s)ds + \int_{y_0}^{y} B(t)dt = 0 \qquad (1.4)$$

where *s* and *t* are variables of integration.

## 2. Homogeneous Equations:

**2.1 Definition:** A differential equation in standard form

$$\frac{dy}{dx} = f(x, y) \tag{2.5}$$

is homogeneous if $f(tx, ty) = f(x, y)$ for every real number $t \neq 0$.
Consider $x \neq 0$. Then, we can write $f(x, y)=f(x, xy/x)=f(1, y/x):=g(y/x)$ for a function $g$ depending only on the ratio $y/x$ .

**2.2 Solution:** The homogeneous differential equation can be transformed into a separable equation by making the substitution:

$$y = xv \tag{2.6}$$

along with its corresponding derivative:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}. \tag{2.7}$$

Then we obtain $v + x\frac{dv}{dx} = g(v)$. This can be rewritten as $\frac{dv}{g(v) - v} = \frac{dx}{x}$ if $g(v) \neq v$.
The resulting equation in the variables $v$ and $x$ is solved as a separable differential equation; the required solution to Equation (2.5) is obtained by back substitution.
The case $g(v) = v$ yields another solution of the form $y = kx$ for any constant $k$.

**Example:** Solve $y' = \frac{y + x}{x}$ for $x \neq 0$.
This differential equation is not separable. Instead it has the form $y' = f(x,y)$,
with $f(x,y) = \frac{y + x}{x}$, where $f(tx, ty) = \frac{ty + tx}{tx} = \frac{y + x}{x} = f(x, y)$,
so it is homogeneous. Substituting equations (2.6) and (2.7) into the equation, we obtain

$$v + x\frac{dv}{x} = \frac{xv + x}{x}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{1}{x}dx - dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}dx - \int dv = c$$

which, when evaluated, yields $v = \ln|x| - c$, or

$$v = \ln|kx| \tag{26}$$

where we have set $c = -\ln|k|$; and have noted that $\ln|x| + \ln|k| = \ln|xk|$.
Finally, substituting $v = y/x$ back into (26), we obtain the solution to the given differential equation as $y = x\ln|kx|$.

## 3. Exact equations

**3.1 Definition:** A differential equation in differential form

$$M(x, y)dx + N(x, y)dy = 0 \tag{27}$$

is exact if there exists a function g(x, y) such that

$$dg(x, y) = M(x, y)dx + N(x, y)dy \tag{28}$$

**3.2 Test for exactness:** If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the $xy$-plane, then Equation (27) is exact if and only if $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$.

**3.3 Solution:** To solve Equation (27), assuming that it is exact, first solve the equations

$$\frac{\partial g(x,y)}{\partial x} = M(x,y) \tag{28}$$

$$\frac{\partial g(x,y)}{\partial y} = N(x,y) \tag{29}$$

for $g(x, y)$. The solution to (27) is then given implicitly by

$$g(x,y) = c \tag{30}$$

where c represents an arbitrary constant.
Equation (30) is immediate from Equations (26) and (27). If (27) is substituted into (26), we obtain $dg(x, y(x)) = 0$. Integrating this equation (note that we can write 0 as 0dx), we have $\int dg(x, y(x)) = \int 0dx$, which, in turn, implies (30).

**Example:** Solve $2xydx + (1 + x^2)dy = 0$.
This equation has the form of Equation (26) with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$. Since $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = 2x$, the differential equation is exact. Because this equation is exact, we now determine a function $g(x, y)$ that satisfies Equations (2.28) and (2.29). Substituting $M(x, y) = 2xy$ into (2.28), we obtain $\frac{\partial g(x,y)}{\partial x} = 2xy$. Integrating both sides of this equation with respect to x, we find

$$g(x,y) = x^2y + h(y) \tag{31}$$

Note that when integrating with respect to x, the constant (with respect to x) of integration can depend on y. We now determine h(y). Differentiating (31) with respect to y, we obtain

$$\frac{\partial g(x,y)}{\partial y} = x^2 + h'(y).$$

Substituting this equation along with $N(x, y) = 1 + x^2$ into (29), we have

$$x^2 + h'(y) = 1 + x^2 \text{ or } h'(y) = 1.$$

Integrating this last equation with respect to y, we obtain $h(y) = y + c_1$ ($c_1$ = constant). Substituting this expression into (31) yields $g(x, y) = x^2y + y + c_1$. The solution to the differential equation, which is given implicitly by (30) as $g(x, y) = c$, is $x^2y + y = c_2$ ($c_2 = c - c_1$). Solving for y explicitly, we obtain the solution as $y = c_2/(x^2 + 1)$.

**3.4 Integrating Factors:**

In general, Equation (27) is not exact. Occasionally, it is possible to transform (27) into an exact differential equation by a judicious multiplication. A function $I(x, y)$ is an integrating factor for (27) if the equation

$$I(x, y)[M(x, y)dx + N(x, y)dy] = 0 \tag{32}$$

is exact.

A solution to (27) is obtained by solving the exact differential equation defined by (32). Some of the more common integrating factors are displayed in Table 2.1 and the conditions that follow:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\equiv g(x)$, a function of $x$ alone, then

$$I(x,y)=e^{\int g(x)dx}$$

If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\equiv h(y)$, a function of $y$ alone, then

$$I(x,y)=e^{-\int h(y)dy}$$

If $M = yf(xy)$ and $N = xg(xy)$, then

$$I(x,y)=\frac{1}{xM-yN}$$

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

**Example:** Solve $ydx - xdy = 0$.
This equation is not exact. It is easy to see that an integrating factor is $I(x)=1/x^2$. Therefore, we can rewrite the given differential equation as

$$\frac{xdy-ydx}{x^2}=0$$

which is exact. This equation can be solved using the steps described in equations (28) through (30).
Alternatively, we can rewrite the above equation as $d(y/x) = 0$. Hence, by direct integration, we have $y / x = c$, or $y = cx$, as the solution.

## 4. Linear Equations

**4.1 Definition:** A first-order *linear* differential equation has the form

$$y' + p(x)y = q(x). \qquad (33)$$

**4.2 Method of Solutions:** An integrating factor for Equation (2.33) is

$$I(x)=e^{\int p(x)dx} \qquad (34)$$

which depends only on $x$ and is independent of $y$. When both sides of (33) are multiplied by $I(x)$, the resulting equation

$$I(x)\,y' + I(x)p(x)y = q(x)I(x) \qquad (35)$$

is exact. This equation can be solved by the method described previously.
A simpler procedure is to rewrite (23) as

$$\frac{d(Iy)}{dx}=Iq\,,$$

and integrate both sides of this last equation with respect to $x$, then solve the resulting equation for $y$. The general solution for Equation (33) is

$$y(x)=e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}q(x)dx+C\right) \qquad (36)$$

**Example:** Solve $y'+(4/x)y=x^4$.
Using (36) for $p(x)=4/x$ and $q(x)=x^4$, we obtain the general solution of the given equation as

$$y = \frac{C}{x^4} + \frac{x^5}{9}.$$

**Table 2.1**

| Group of terms | Integrating factor $I(x, y)$ | Exact differential $dy(x, y)$ |
|---|---|---|
| $y\,dx - x\,dy$ | $-\frac{1}{x^2}$ | $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$ |
| $y\,dx - x\,dy$ | $\frac{1}{y^2}$ | $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$ |
| $y\,dx - x\,dy$ | $-\frac{1}{xy}$ | $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$ |
| $y\,dx - x\,dy$ | $-\frac{1}{x^2+y^2}$ | $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\arctan\frac{y}{x}\right)$ |
| $y\,dx + x\,dy$ | $\frac{1}{xy}$ | $\frac{y\,dx + x\,dy}{xy} = d(\ln xy)$ |
| $y\,dx + x\,dy$ | $\frac{1}{(xy)^n}$, $n > 1$ | $\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$ |
| $y\,dy + x\,dx$ | $\frac{1}{x^2+y^2}$ | $\frac{y\,dy + x\,dx}{x^2+y^2} = d\left[\frac{1}{2}\ln(x^2+y^2)\right]$ |
| $y\,dy + x\,dx$ | $\frac{1}{(x^2+y^2)^n}$, $n > 1$ | $\frac{y\,dy + x\,dx}{(x^2+y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$ |
| $ay\,dx + bx\,dy$ ($a$, $b$ constants) | $x^{a-1}y^{b-1}$ | $x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$ |

## 5. Bernoulli Equations

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^{\alpha} \quad (37)$$

where $\alpha$ is a real number ($\alpha \neq 0$; $\alpha \neq 1$). If $\alpha > 0$, then $y \equiv 0$ is a solution of (37). Otherwise, if $\alpha < 0$, then the condition is $y \neq 0$. In both cases, we now find the solutions $y \neq 0$. To do this we divide both sides by $y^{\alpha}$ to obtain $y^{-\alpha}y' + p(x)\,y^{1-\alpha} = q(x)$. The substitution $z = y^{1-\alpha}$ now transforms (37) into a linear differential equation in the unknown function $z(x)$.

**Example:** Solve $y' + xy = xy^2$.

This equation is not linear. It is, however, a Bernoulli differential equation having the form of Equation (37) with $p(x) = q(x) = x$, and $\alpha = 2$. First, we can see that $y \equiv 0$ is a solution of the equation. We now find the solution $y \neq 0$. To do so, we make the substitution: $z = y^{1-2} = y^{-1}$, from which follow $y = 1/z$ and $y' = -z'/z^2$. Substituting these equations into the given differential equation, we obtain the equation $z' - xz = -x$ which is linear for the unknown function $z(x)$. It has the form of Equation (2.33) with $y$ replaced by $z$ and $p(x) = q(x) = -x$. Using the formula (2.36) we obtain that

$$z = Ce^{\frac{x^2}{2}} + 1.$$

The solution of the original differential equation is then: $y = \frac{1}{z} = \frac{1}{Ce^{\frac{x^2}{2}} + 1}$.

## 6. Modelling: Electric Circuits

Differential equations are of interest to non-mathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem.
These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive in an experimental setting. Nevertheless, mathematical modelling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models a re validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and may indicate fairly precisely what experimental data will be most helpful. In Section 1.1 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in that section. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modelling.

***6.1 Construction of the Model.*** This involves a translation of the physical situation into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton’s laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process. It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton’s laws, insect populations do not grow indefinitely as stated because of eventual limitations on their food supply, and heat transfer is affected by factors other than the temperature difference. Alternatively, one can adopt the point of view that the mathematical equations exactly describe the operation of a simplified physical model, which has been constructed (or conceived of) so as to embody the most important features of the actual process. Sometimes, the process of mathematical modelling involves the conceptual replacement of a discrete process by a continuous one. For instance, the number of members in an insect population changes by discrete amounts; however, if the population is large, it seems reasonable to consider it as a continuous variable and even to speak of its derivative.

***6.2 Analysis of the Model.*** Once the problem has been formulated mathematically, one is often faced with the problem of solving one or more differential equations or, failing that, of finding out as much as possible about the properties of the solution. It may happen that this mathematical problem is quite difficult and, if so, further approximations may be indicated at this stage to make the problem mathematically tractable. For example, a nonlinear equation may be approximated by a linear one, or a slowly varying coefficient may be replaced by a constant. Naturally, any such approximations must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process under investigation. At the same time, an intimate knowledge of the physics of the problem may suggest reasonable mathematical approximations that will make the mathematical problem more amenable to analysis. This interplay of understanding of physical phenomena and knowledge of mathematical techniques and their limitations is characteristic of applied mathematics at its best, and is indispensable in successfully constructing useful mathematical models of intricate physical processes.

***6.3 Comparison with Experiment or Observation.*** Finally, having obtained the solution (or at least some information about it), you must interpret this information in the context in which the problem arose. In particular, you should always check that the mathematical solution appears physically reasonable. If possible, calculate the values of the solution at selected points and compare them with experimentally observed values. Or, ask whether the behavior of the solution after a long time is consistent with observations. Or, examine the solutions corresponding to certain special values of parameters in the problem. Of course, the fact that the mathematical solution appears to be reasonable does not guarantee it is correct. However, if the predictions of the mathematical model are seriously inconsistent with observations of the physical system it purports to describe, this suggests that either errors have been made in solving the mathematical problem, or the mathematical model itself needs refinement, or observations must be made with greater care. In Chapter 1 we have given some examples which are typical of applications in which first-order differential equations arise. In this section we pay our attention to a concrete model, that is a mathematical model of electric circuits. We start with some important facts from electric circuits.

**6.4 Electric circuits.** The simplest electric circuit is a series circuit in which we have a source of electric energy (**electromotive force**) such as a generator or a battery, and a resistor, which uses the energy. Experiments show that the following law holds.

*The* ***voltage drop*** $E_R$ ***across a resistor*** *is proportional to the instantaneous the current I,* say,

**(Ohm's law)**
$$E_R = RI\,, \tag{L1}$$

where the constant of proportional $R$ is call the resistance of the resistor. The current $I$ is measured in *amperes*, the resistance $R$ in *Ohms*, and the voltage $E_R$ in *volts*.

The other two important elements in more complicated circuits are inductors and capacitors. An inductor opposes a change in current, having an inertia effect in electricity similar to that of mass in mechanics; we shall consider this analogy latter. Experiments yield the following law.

*The* ***voltage drop*** $E_R$ ***across an inductor*** *is proportional to the instantaneous time rate of change of the current I,* say,

$$E_L = L\frac{dI}{dt} \tag{L2}$$

where the constant of proportional $L$ is called the **inductance** of the inductor and is measured in *henrys*; time t is measured in seconds.

A capacitor is an element which stores energy. Experiments yield the following law.

*The* ***voltage drop*** $E_C$ ***across an capacitor*** *is proportional to the instantaneous electric charge Q on the capacitor,* say,

$$E_C = \frac{1}{C}Q \tag{L3*}$$

where $C$ is called the **capacitance** and is measured in *farads;* the charge Q is measured in *coulombs*. Since

$$I(t) = \frac{dQ}{dt} \qquad \text{(L3')}$$

this may be written

$$E_C = \frac{1}{C}\left[Q(t_0) + \int_{t_0}^{t} I(\tau)d\tau\right] \qquad \text{(L3)}$$

The current I(t) in a circuit may be determined by solving the equation (or equations) resulting from the application of the following law.

**6.5 Kirchhoff's voltage law (KVL):**
*The algebraic sum of all the instantaneous voltage drops around any closed loop is zero, or the voltage impressed on a closed loop is equal to the sum of the voltage drops in the rest of the loop.*

**6.6 Example: RL-circuit**

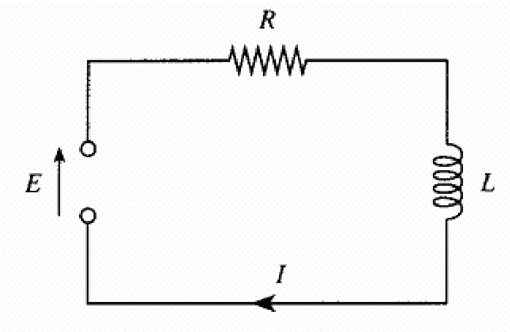



Fig. 2.2

Model the "RL-circuit" in fig 2.2 and solve the resulting equation for: (A) a constant electromotive force; (B) a period electromotive force.

**Solution: 1st Step. Modeling.** By (L1) the voltage drop across the resistor is *RI*. By (L2) ) the voltage drop across the inductor is *LdI/dt*. By KVL the sum of the two voltage drops must equal the electromotive force *E(t)*; thus

$$L\frac{dI}{dt} + RI = E(t)$$

**2nd Step. Solution of the equation.** In order to use the formula (2.36) we transform the above equation to the standard form by deviding both side to $L$ and obtain

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L}.$$

Using now formula (2.36) with x=$t$, $y=I$, $p=R/L$, and $q=E/L$ we get

$$I(t) = e^{-\alpha t}\left[\int e^{\alpha t}\frac{E(t)}{L}dt + c\right] \quad \text{for } \alpha = R/L.$$

**3rd Step. Case A: Constant electromotive force *$E=E_0$*.** The above equality for *I(t)* yields

$$I(t) = e^{-\alpha t}\left[\frac{E_0}{R}e^{\alpha t} + c\right] = \frac{E_0}{R} + ce^{-\alpha t}$$

The last term tends to zero as t→∞; practically, after some time the current *I(t)* will be constant, equal to $E_0/R$, the value it would have immediately (by Ohm's law) had we no inductor in the circuit, and we see that this limit is independent of the initial value *I(0)*.

**Case B: Periodic electromotive force $E=E_0$ Sin$\omega t$.** For this *E(t)* we have that

$$I(t)=e^{-\alpha t}\left[\frac{E_0}{L}\int e^{\alpha t}\sin\omega t dt+c\right] \quad \text{for } \alpha = R/L.$$

Integration by part yields

$$I(t)=ce^{-\alpha t}+\frac{E_0}{\sqrt{R^2+\varpi^2L^2}}\sin(\omega t-\delta), \text{ for } \delta=\arctan(\omega L/R).$$

The exponential term will approach zero as *t* tends to infinity. This mean that after some time the current *I(t)* will execute practically harmonic oscillations.

### 7. Existence and Uniqueness Theorem

We now finish this chapter by stating the theorem on existence and uniqueness of the solution of an initial-value problem for a first-order differential equation.

**7.1 Theorem.**
Let the functions *f(t,y)* and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0,y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \varphi(t)$ of the initial value problem
$y' = f(t, y), \quad y(t_0) = y_0$.

## Problems

**I.** In each of Problems 1 through 8 solve the given differential equation.

1. $y' = x^2/y$
2. $y' = x^2/y(1+x^3)$
3. $y' + y^2\sin x = 0$
4. $y' = (3x^2-1)/(3+2y)$
5. $y' = (\cos^2 x)(\cos^2 2y)$
6. $xy' = (1-y^2)^{1/2}$
7. $\dfrac{dy}{dx} = \dfrac{x-e^{-x}}{y+e^y}$
8. $\dfrac{dy}{dx} = \dfrac{x^2}{1+y^2}$

**II.** In each of Problems 9 through 20 find the solution of the given initial value problem in explicit form.

9. $y' = (1-2x)y^2, \quad y(0) = -1/6$
10. $y' = (1-2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = 2x/(y+x^2y), \quad y(0) = -2$
14. $y' = xy^3(1+x^2)^{-1/2}, \quad y(0) = 1$
15. $y' = 2x/(1+2y), \quad y(2) = 0$
16. $y' = x(x^2+1)/4y^3, \quad y(0) = -1/\sqrt{2}$
17. $y' = (3x^2-e^x)/(2y-5), \quad y(0) = 1$
18. $y' = (e^{-x}-e^x)/(3+4y), \quad y(0) = 1$
19. $\sin 2x\,dx + \cos 3y\,dy = 0, \quad y(\pi/2) = \pi/3$
20. $y^2(1-x^2)^{1/2}dy = \arcsin x\,dx, \quad y(0) = 0$

**III.** In each of Problems 31 through 38:
(a) Show that the given equation is homogeneous.
(b) Solve the differential equation.

31. $\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}$

32. $\frac{dy}{dx}=\frac{x^2+3y^2}{2xy}$

33. $\frac{dy}{dx}=\frac{4y-3x}{2x-y}$

34. $\frac{dy}{dx}=-\frac{4x+3y}{2x+y}$

35. $\frac{dy}{dx}=\frac{x+3y}{x-y}$

36. $(x^2+3xy+y^2)dx-x^2dy=0$

37. $\frac{dy}{dx}=\frac{x^2-3y^2}{2xy}$

38. $\frac{dy}{dx}=\frac{3y^2-x^2}{2xy}$

**IV.** Determine whether or not each of the equations in Problems 1 through 12 is exact. If it is exact, Find the solution.

1. $(2x+3)+(2y-2)y'=0$
2. $(2x+4y)+(2x-2y)y'=0$
3. $(3x^2-2xy+2)\,dx+(6y^2-x^2+3)\,dy=0$
4. $(2xy^2+2y)+(2x^2y+2x)y'=0$
5. $\frac{dy}{dx}=-\frac{ax+by}{bx+cy}$
6. $\frac{dy}{dx}=-\frac{ax-by}{bx-cy}$
7. $(e^x\sin y-2y\sin x)\,dx+(e^x\cos y+2\cos x)\,dy=0$
8. $(e^x\sin y+3y)\,dx-(3x-e^x\sin y)\,dy=0$
9. $(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)\,dx+(xe^{xy}\cos 2x-3)\,dy=0$
10. $(y/x+6x)\,dx+(\ln x-2)\,dy=0,\qquad x>0$
11. $(x\ln y+xy)\,dx+(y\ln x+xy)\,dy=0;\qquad x>0,\quad y>0$
12. $\frac{x\,dx}{(x^2+y^2)^{3/2}}+\frac{y\,dy}{(x^2+y^2)^{3/2}}=0$

In each of Problems 13 and 14 solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x-y)\,dx+(2y-x)\,dy=0,\qquad y(1)=3$
14. $(9x^2+y-1)\,dx-(4y-x)\,dy=0,\qquad y(1)=0$

In each of Problems 15 and 16 find the value of $b$ for which the given equation is exact and then solve it using that value of $b$.

15. $(xy^2+bx^2y)\,dx+(x+y)x^2\,dy=0$
16. $(ye^{2xy}+x)\,dx+bxe^{2xy}\,dy=0$

**V.** Show that the equations in Problems 1 through 2 are not exact, but become exact when multiplied by the given integrating factor. Then solve the equations.

1. $x^2y^3+x(1+y^2)y'=0,\qquad \mu(x,y)=1/xy^3$
2. $\left(\frac{\sin y}{y}-2e^{-x}\sin x\right)dx+\left(\frac{\cos y+2e^{-x}\cos x}{y}\right)dy=0,\qquad \mu(x,y)=ye^x$
3. $y\,dx+(2x-ye^y)\,dy=0,\qquad \mu(x,y)=y$
4. $(x+2)\sin y\,dx+x\cos y\,dy=0,\qquad \mu(x,y)=xe^x$

**VI.** Show that if $(N'_x-M'_y)/(xM-yN)=G$, where $G$ depends on the quantity $xy$ only, then the differential equation $Mdx+Ndy=0$ has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

**VII.** In each of Problems 1 through 5 find an integrating factor and solve the given equation.

1. $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$

2. $y'=e^{2x}+y-1$

3. $dx + (x/y - \sin y)dy = 0$

4. $y\,dx + (2xy - e^{-2y})dy = 0$

5. $e^x dx + (e^x \cot y + 2y \csc y)dy = 0$

6. $[4(x^3/y^2) + (3/y)]\,dx + [3(x/y^2) + 4y]\,dy$

**VIII**. In each of Problems 1 through 12 find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2 e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 3\cos 2t, \quad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \quad t > 0$
7. $y' + 2ty = 2te^{-t^2}$
8. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^2 e^{-t}$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

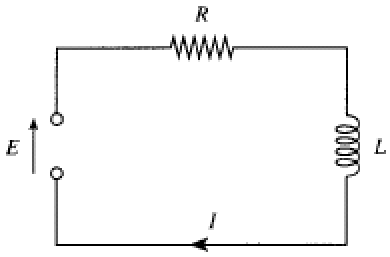
In each of Problems 13 through 20 find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \quad y(0) = 1$
14. $y' + 2y = te^{-2t}, \quad y(1) = 0$
15. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2 \quad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \quad y(0) = 2$
18. $ty' + 2y = \sin t, \quad y(\pi/2) = 1$
19. $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$
20. $ty' + (t+1)y = t, \quad y(\ln 2) = 1$

**IX.** In each of Problems 28 through 31 solve the given Bernoulli equation:

28. $t^2 y' + 2ty - y^3 = 0, \quad t > 0$
29. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.
30. $y' = \epsilon y - \sigma y^3$, $\epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.
31. $dy/dt = (\Gamma \cos t + T)y - y^3$, where $\Gamma$ and $T$ are constants. This equation also occurs in the study of the stability of fluid flow.

**X.** Consider RL-circuit



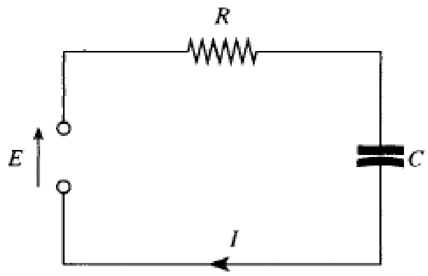

**(a)** Determine the differential equation governing the current $I$ (in amperes) on the circuit.

**(b)** Solve the equation to find the current in the case of constant electromotive force $E(t)=E$, constant. Evaluate the constant of integration by using the condition $I(0)=I_0$.

**(c)** Determine the limit $\lim_{t\to\infty} I(t)$ where $I(t)$ is obtained from **(b)**

**(d)** Let $R$= 100 ohms, $L$=2.5 henries, $E(t) = 110\cos 314t$. Find the steady-state solution.

**XI.** Consider RC-circuit




(a) Determine the differential equation governing the amount of electrical charge $q$ (in coulombs) on the capacitor.
(b) Solve the equation to find the charge $q$ in the case of constant electromotive force $E(t)=E$, constant. Evaluate the constant of integration by using the condition $q(0)=q_0$.
(c) Determine the limit $\lim_{t\to\infty}q(t)$.

# CHAPTER 5: SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second-order linear differential equations. As an example, we discuss the oscillations of some basic mechanical and electrical systems at the end of the chapter.

## 1. Definitions and Notations

A second order ordinary differential equation has the form

$$y''(t)=f(t,y,y') \qquad \text{(DE2)}$$

where $f$ is some given function. Usually, we will denote the independent variable by $t$ since time is often the independent variable in physical problems, but sometimes we will use $x$ instead. Equation (DE2) is said to be linear if the function $f$ has the form

$$f(t, y, y')= r(t) - q(t)y - p(t)y'$$

that is, if $f$ is linear in $y$ and $y'$. In this expression, $r$, $p$, and $q$ are specified functions of the independent variable $t$ but do not depend on $y$. In this case we usually rewrite Eq. (DE2) as $y'' + p(t)y' + q(t)y = r(t)$, and make the following precise definition.

**1.1 Definition.** A second-order differential equation is called **linear** if it can be written in the form

$$y'' + p(t)y' + q(t)y = r(t). \qquad (1)$$

Instead of Eq. (1), we often see the equation

$$P(t)y'' + Q(t)y' + G(t)y = R(t).$$

Of course, if $P(t) \neq 0$, we can divide this Eq. by $P(t)$ and thereby obtain Eq. (1) with $p(t) =Q(t)/P(t)$, $q(t) =G(t)/P(t)$, $r(t) =R(t)/P(t)$.

In discussing Eq. (1) and in trying to solve it, we will restrict ourselves to intervals in which $p$, $q$, and $r$ are continuous functions on some open interval $I$, that is for $\alpha < t < \beta$. The cases $\alpha = -\infty$, or $\beta =\infty$, or both, are included. The function $p(t)$ and $q(t)$ are called the **coefficients** of the Eq. (1).

A second-order linear equation is said to be **homogeneous** if the term $r(t)$ in Eq. (1), is zero for all $t$. Otherwise, the equation is called **nonhomogeneous**. As a result, the term $r(t)$ is sometimes called the nonhomogeneous term.

**Examples:** The following equations: $y''+4y=e^{-t}\sin t$ and $y''-2ty'+6(1-t^2)y=0$ are examples of nonhomogeneous and homogeneous second-order linear equations, respectively.

**1.2 Solution.** A solution of a second-order linear differential equation on some interval $I$ is a function $y=h(t)$ that is twice differentiable and satisfies the differential equation for all $t\in I$.

## 2. Theory for Solutions of Linear Homogeneous Equations

In this section we give a general theory for solutions to Linear Homogeneous Equations on some interval $I$ of the form

$$y'' + p(t)y' + q(t)y = 0. \quad (2)$$

with continuous coefficients $p$, $q$ and initial conditions

$$y(t_0)=y_0;\ y'(t_0)=y_1 \text{ for given } y_0, y_1, \text{ and some fixed } t_0\in I. \quad (3)$$

We accept the following theorem on existence and uniqueness theorem of the solution to the initial-value problem (2), (3).

**2.1 Existence and Uniqueness Theorem:** *Consider the initial-value problem*

$$y'' + p(t)y' + q(t)y = r(t),\ y(t_0) = y_0,\ y'(t_0) = y_1 \quad (IVP)$$

*where p, q, and r are continuous on an open interval I . Then there is exactly one solution $y = \varphi(t)$ of this problem, and the solution exists throughout the interval I.*

**2.3 Linearity Principle.** If $y_1$ and $y_2$ are two solutions of the differential equation (2), then the linear combination $c_1y_1 + c_2 y_2$ is also a solution for any values of the constants $c_1$ and $c_2$.

PROOF. The assertion follows from the following direct substitution:
$(c_1y_1 + c_2 y_2)''+p(c_1y_1 + c_2 y_2)'+q(c_1y_1 + c_2 y_2)= c_1(y_1'' + py_1'+qy_1)+ c_2(y_2'' + py_2'+qy_2) = 0.$

**2.4 Linear Independence of Solutions:**
The two solutions $y_1$ and $y_2$ are called **linearly independent** on $I$ if

$$k_1 y_1(t) + k_2 y_2(t)=0 \text{ for all } t\in I \text{ implies } k_1 = k_2=0;$$

and we call $y_1$, $y_2$ **linearly dependent** on $I$ if there exist $k_1$, $k_2$ not both zero such that $k_1 y_1(t) + k_2 y_2(t)=0$ for all $t\in I$. In this case (and only in this case) $y_1$, $y_2$ are proportional, that is $y_1=ky_2$ if $k_1\neq 0$, or $y_2=ly_1$ if $k_2\neq 0$. Since, $y_1$ and $y_2$ are linearly independent if and only if they are not linearly dependent, we obtain that, two solutions $y_1$ and $y_2$ are linearly independent if and only if they are not proportional.

**Example.** $y_1(t) =e^t$ and $y_2(t)=e^{-2t}$ are linearly independent, because they are not proportional.

The following notion of Wronski determinant is very helpful in characterizing the linear independence of solutions.

**2.5 Definition.** The **Wronski Determinant** (or **Wronskian**) of the two solutions $y_1$, $y_2$ of the equation (2) is defined by

$$W(y_1, y_2)= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2'- y_1'y_2$$

The following theorems connect the linear dependence and independence of the two solutions of Eq. (2) with the properties of their Wronskian.

**2.6 Theorem.** *Two solutions $y_1$ and $y_2$ of Eq. (2) are linearly dependent on I if and only if their Wronskian $W(y_1, y_2)$ is zero at some point $t_0 \in I$.*

PROOF. If $y_1$ and $y_2$ are linearly dependent on $I$, then they are proportional, say, $y_1 = k\, y_2$ on $I$. This follows that $W(y_1, y_2)(t)=0$ for all $t \in I$.
Conversely, if there is $t_0 \in I$ such that $W(f,g)(t_0)=0$, we prove that $f$ and $g$ are linearly dependent on $I$. In fact, since $W(y_1, y_2)(t_0) = 0$ we have that the system of equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0 \end{aligned} \qquad (4)$$

for the unknowns $c_1$ and $c_2$ has a nontrivial solution. Using these values of $c_1$ and $c_2$,
let $\varphi(t) = c_1 y_1(t) + c_2\, y_2(t)$. Then $\varphi$ is a solution of Eq. (2), and by system (3) $\varphi$ also satisfies the initial conditions

$$\varphi(t_0) = 0,\ \varphi'(t_0) = 0.$$

Therefore, by the existence and uniqueness theorem 2.1, $\varphi(t) = 0$ for all $t$ in $I$. Since $\varphi(t) = c_1 y_1(t) + c_2 y_2(t)$ with $c_1$ and $c_2$ not both zero, this means that $y_1$ and $y_2$ are linearly dependent.

**Remark:** The above proof also shows that two solutions $y_1$ and $y_2$ of Eq. (2) are linearly dependent on $I$ if and only if their Wronskian $W(y_1, y_2)$ is zero for all $t \in I$.

**2.7 Theorem.** *Let $y_1$ and $y_2$ be two solutions of the equation (2) on an interval I . Then, the following assertions are equivalent.*

*(i)* $y_1$ *and* $y_2$ *are linearly independent.*
*(ii)* $W(y_1, y_2)(t_0) \neq 0$ *for some point* $t_0$ *in I.*
*(iii)* $W(y_1, y_2)(t) \neq 0$ *for every t in I.*

PROOF. "(i) $\Rightarrow$ (ii)": For the purpose of contradiction let $W(y_1, y_2)(t) =0$ for all $t$ in $I$. Then, by Theorem 2.6, $y_1$ and $y_2$ are linearly dependent. This contradicts to (i).

"(ii) $\Rightarrow$ (iii)": Again, for the purpose of contradiction suppose that $W(y_1, y_2)(t_1) =0$ for some $t_1$ in $I$. Then, by theorem 2.6, $y_1$ and $y_2$ are linearly dependent. This yields that $W(y_1, y_2)$ is zero for all $t \in I$ (see Remark after theorem 2.6). This contradicts to (ii).

"(iii) $\Rightarrow$ (i)": If $y_1$ and $y_2$ are linearly dependent, then, by theorem 2.6, there exists $t_0$ such that $W(y_1, y_2)(t_0) =0$. This contradicts to (iii).

**2.8 Theorem (Existence of Linearly Independent Solutions**).
*Consider Eq. (2) with continuous coefficients p, q on I. Then there exists two linearly independent solutions $y_1, y_2$ on I of Eq. (2).*

PROOF. By theorem 2.1, there exists solution $y_1$ of Eq. (2) satisfying $y_1(t_0)=1$, $y_1'(t_0)=0$ for some $t_0$ in $I$. Also, there exists solution $y_2$ of Eq. (2) satisfying $y_2(t_0)=0$, $y_2'(t_0)=1$. Therefore, $W(y_1, y_2)(t_0)=1 \neq 0$. Hence, $y_1$ and $y_2$ are linearly independent.

**2.9 Theorem (Structure of Solutions to Homogeneous Equations).**
*Consider Eq. (2) with continuous coefficients p, q on I. Let $y_1, y_2$ be two linearly independent solutions on I of Eq. (2). Then the general solution (the set of all solutions) of Eq. (2) is*

$$\{c_1\, y_1 + c_2\, y_2 \mid c_1 \text{ and } c_2 \text{ are arbitrary constants}\} \qquad (5)$$

PROOF. Clearly, for any fixed constants $c_1$ and $c_2$, the formula in (5) represents a solution of Eq. (2) on $I$. Let now $Y$ be an arbitrary solution of (2) on $I$. Put $Y(t_0)=k_1$, $Y'(t_0)=k_2$ for some fixed $t_0$ in $I$.

Consider y=$c_1 y_1+c_2 y_2$. We will find $c_1$, $c_2$ such that $y(t_0)=k_1$, $y'(t_0)=k_2$. These conditions are equivalent to

$$\begin{cases} c_1y_1(t_0) + c_2y_2(t_0) = k_1 \\ c_1y_1'(t_0) + c_2y_2'(t_0) = k_2 \end{cases} \quad (6)$$

This system of linear equations (6) has a solution ($c_{10}$ , $c_{20}$) since its determinant of coefficients is $W(y_1,y_2)(t_0) \neq 0$ (because $y_1$, $y_2$ are linearly independent).
The theorem 2.1 now yields that $Y(t)= c_{10}\, y_1(t)+c_{20}\, y_2(t)$ for all $t$ in $I$. Therefore, $Y$ has a form represented in the set (5).

**2.10 Definition.** A **basis (**or a **fundamental set) of solutions** of Eq. (2) on an interval $I$ is a pair $y_1$, $y_2$ of linearly independent solutions on $I$ of Eq. (2).

**Remark.**

(i) To solve the homogeneous equation (2) is to find a basic of solutions $y_1$,$y_2$ of Eq. (2). Then, the general solution is y=$c_1 y_1+c_2 y_2$.

(ii) However, for most problems of the form (2), it is not possible to write down a useful expression for the solution. This is a major difference between first order and second order linear equations.

**Example.** Solve $y''-3y'+2y=0$ (7)

To solve this equation, we remember from chapter 2 that a first-order linear differential equation y'+ky=0 with constant coefficient k has an exponential function as a solution, y=$e^{-kt}$. This gives us the idea to try as a solution of (7) the function

$$y=e^{ct}. \quad (8)$$

Substituting (8) into (7) we obtain ($c^2$-3c+2) $e^{ct}$=0, this is equivalent to c=1 or c=2. We then obtain the two following linearly independent solutions of (7): $y_1= e^t$ and $y_2= e^{2t}$. Therefore, the general solution of (7) is y= $c_1 e^t +c_2e^{2t}$

**2.11. Reduction of Order.**

If a nontrivial solution $y_1$ is known, then we can find the solution $y_2$ linearly independent with $y_1$ by the following procedure which is called the method of reduction of order.

Putting $y_2$=u. $y_1$ and substituting it to Eq (2), we obtain

$$u''\, y_1 +u'(2\, y_1'+p\, y_1)=0.$$

Setting U=u', it follows that U' $y_1$+U(2 $y_1$'+$p$ $y_1$)=0, this yields $U=\frac{e^{-\int p dt}}{y_1^2}$. Returning to u we have that $u'=\left(\frac{y_2}{y_1}\right)' = \frac{e^{-\int p dt}}{y_1^2}$. Therefore,

$$y_2 = y_1 \int \frac{e^{-\int p dt}}{y_1^2} dt \quad (9)$$

**Example:** Solve $(t^2-1)y''-2ty'+2y=0$ given a solution $y_1$=t.

To use the formula (9) we write the equation in the standard form

$$y''-2ty'/(t^2-1)+2y/(t^2-1)=0.$$

Applying (9) we have $y_2 = t\int \frac{e^{\int \frac{2t}{t^2-1}dt}}{t^2} dt = t^2+1$. Therefore, the general solution of given equation is $y=c_1t+c_2(t^2+1)$.

## 3. Homogeneous Equations with Constant Coefficients

In this section we consider the homogeneous equations with constant coefficients of the form

$$y'' + ay' + by = 0 \qquad (10)$$

which has arbitrary (real) constant coefficients. Based on our experience with Eq. (7), let us also seek exponential solutions of Eq. (10). Thus we suppose that $y = e^{kt}$, where $k$ is a parameter to be determined. Then it follows that $y' = ke^{kt}$ and $y'' = k^2e^{kt}$. By substituting these expressions for $y$, $y''$, and $y''$ in Eq. (10), we obtain
$(k^2 + ak + b)e^{kt} = 0$ or, since $e^{kt}$ is never zero,

$$k^2 + ak + b = 0. \qquad (11)$$

Equation (11) is called the **characteristic equation** for the differential equation (10). Its significance lies in the fact that if $k$ is a root of the polynomial equation (11), then $y = e^{kt}$ is a solution of the differential equation (10). Since Eq. (11) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates. We now consider each case in detail.

**1st Case: Distinct real roots.**

Assuming that the roots of the characteristic equation (11) are real and different, let them be denoted by $k_1$ and $k_2$, where $k_1 \neq k_2$. Then $y_1(t) = e^{k_1t}$ and $y_2(t) = e^{k_2t}$ are two linearly dependent solutions of Eq. (11). Therefore, by Theorem 2.9 in the previous section, we obtain the general solution of Eq. (11):

$$y = c_1 e^{k_1t} + c_2 e^{k_2t} \qquad (12)$$

**Example.** Find the general solution of

$$y'' + 5y' + 6y = 0. \qquad (13)$$

The characteristic equation is

$$k^2 + 5k + 6 = 0.$$

It has two distinct real roots: $k_1 = -2$ and $k_2 = -3$; then the general solution of Eq. (13) is

$$y = c_1e^{-2t} + c_2e^{-3t}.$$

**2nd Case: Double real root.**

We consider the second possibility, namely, that the two real roots $k_1$ and $k_2$ are equal. This case occurs when the discriminant $\Delta = a^2 - 4b$ is zero, and it follows from the quadratic formula that

$$k_1 = k_2 = -a/2. \qquad (14)$$

The difficulty is immediately apparent; both roots yield the same solution.

$$y_1(t) = e^{k_1t} = e^{-at/2} \qquad (15)$$

of the differential equation (11). We now find a second solution $y_1$ which is linearly independent to $y_2$. Using formula (9) we obtain that

$$y_2 = e^{-\frac{at}{2}} \int \frac{e^{-\int a dt}}{e^{-at}} dt = t\, e^{-\frac{at}{2}}$$

Therefore, the general solution of Eq. (11) in this case is

$$y = (c_1 + c_2t)\, e^{-\frac{at}{2}}.$$

**Example.** Solve the differential equation $y'' + 4y' + 4y = 0$.

The characteristic equation is $k^2 + 4k + 4 = 0$, which has a double real root $k_1 = k_2 = -2$. Therefore, the general solution of given differential equation is $y = (c_1 + c_2t)\, e^{-2t}$.

**3rd Case: Complex conjugate roots.**

Suppose now that $a^2 - 4b$ is negative. Then the roots of characteristic Eq. (3) are conjugate complex numbers; we denote them by $k_1 = \lambda + i\mu$, $k_2 = \lambda - i\mu$, where $\lambda$ and $\mu$ are real and $\mu \neq 0$. We obtain two linearly independent (complex-value) solutions of Eq. (2) as

$$Y_1(t) = e^{(\lambda + i\mu)t}; \quad Y_2(t) = e^{(\lambda - i\mu)t}. \qquad (16)$$

To find real-value solutions we will recall the Euler Formula:

$$e^{it} = \cos t + i \sin t \text{ for every real number } t.$$

Using this formula for $Y_1(t)$, $Y_2(t)$ we obtain

$$\frac{1}{2}[Y_1(t) + Y_2(t)] = \frac{1}{2}(e^{\lambda t}(\cos\mu t + i\sin\mu t) + e^{\lambda t}(\cos\mu t - i\sin\mu t)) = e^{\lambda t}\cos\mu t$$

$$\frac{1}{2i}[Y_1(t) - Y_2(t)] = \frac{1}{2i}(e^{\lambda t}(\cos\mu t + i\sin\mu t) - e^{\lambda t}(\cos\mu t - i\sin\mu t)) = e^{\lambda t}\sin\mu t.$$

Since, a linear combination of two solutions of Eq.(2) is again a solution of Eq. (2), we obtain the two following linearly independent (real-value) solutions of Eq. (2):

$$y_1 = e^{\lambda t}\cos\mu t \text{ and } y_2 = e^{\lambda t}\sin\mu t.$$

Therefore, the general solution of Eq. (2) is

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t.$$

**Example 1.** Find the general solution of

$$y'' + y' + y = 0. \qquad (17)$$

The characteristic equation is $k^2 + k + 1 = 0$, and its roots are $k = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Thus $\lambda = -1/2$ and $\mu = \sqrt{3}/2$, so the general solution of Eq. (17) is

$$y = c_1 e^{-t/2}\cos(\sqrt{3}\,t/2) + c_2 e^{-t/2}\sin(\sqrt{3}\,t/2).$$

**Example 2.** Find the general solution of $y'' + 9y = 0$.

The characteristic equation is $k^2 + 9 = 0$ with the roots $k = \pm 3i$ ; thus $\lambda = 0$ and $\mu = 3$. The general solution is $y = c_1\cos 3t + c_2\sin 3t$.

Note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution.

## 4. Modelling: Free Oscillation (Mass-spring problem)

We will study the motion of a mass on a spring in detail because an understanding of the behaviour of this simple system is the first step in the investigation of more complex vibrating systems. Further, the principles involved are common to many problems. Consider a mass $m$ hanging on the end of a vertical spring of original length $l$, as shown in Figure 4.1. The mass causes an elongation $L$ of the spring in the downward (positive) direction. There are two forces acting at the point where the mass is attached to the spring; see Figure 4.2. The gravitational force, or weight of the mass, acts downward and has magnitude $mg$, where $g$ is the acceleration due to gravity. There is also a force $F_s$ , due to the spring, that acts upward. If we assume that the elongation $L$ of the spring is small, the spring force is very nearly proportional to $L$; this is known as Hooke's law. Thus we write $F_s = -kL$, where the constant of proportionality $k$ is called the spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction.

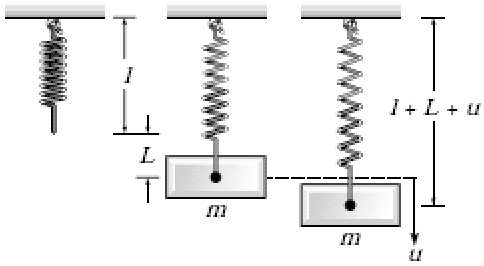



**Fig 4.1** A spring–mass system.

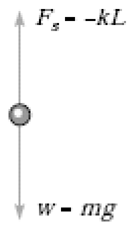



**Fig 4.2** Force diagram for a spring–mass system.

Since the mass is in equilibrium, the two forces balance each other, which means that

$$mg - kL = 0. \quad (1)$$

For a given weight $w = mg$, one can measure $L$ and then use Eq. (2) to determine $k$. Note that $k$ has the units of force/length. In the corresponding dynamic problem we are interested in studying the motion of the mass when it is acted on by an external force or is initially displaced. Let $u(t)$, measured positive downward, denote the displacement of the mass from its equilibrium position at time $t$; see Figure 4.1. Then $u(t)$ is related to the forces acting on the mass through Newton's law of motion,

$$mu''(t) = f(t), \quad (2)$$

where $u''$ is the acceleration of the mass and $f$ is the net force acting on the mass. Observe that both $u$ and $f$ are functions of time. In determining $f$, we consider the following cases.

**4.1 Undamped Systems.** In this case there are two separate forces that must be considered:

**1. The weight** $w = mg$ of the mass always acts downward.
**2. The spring force** $F_S$ is assumed to be proportional to the total elongation $L + u$ of the spring and always acts to restore the spring to its natural position. If $L + u > 0$, then the spring is extended, and the spring force is directed upward. In this case

$$Fs = -k(L + u). \quad (3)$$

On the other hand, if $L + u < 0$, then the spring is compressed a distance $|L + u|$, and the spring force, which is now directed downward, is given by $Fs = k|L + u|$. However, when $L + u < 0$, it follows that $|L + u| = -(L + u)$, so $F_s$ is again given by Eq. (3). Thus, regardless of the position of the mass, the force exerted by the spring is always expressed by Eq. (3).

Taking account of these forces, we can now rewrite Newton's law (2) as

$$mu''(t) = mg + Fs(t) = mg - k[L + u(t)]$$

Since $mg - kL = 0$ by Eq. (1), it follows that the equation of motion of the mass is

$$mu''(t) + ku(t) = 0, \quad (4)$$

where the constants $m$ and $k$ are positive. Note that Eq. (4) has the same form as Eq. (1).
It is important to understand that Eq. (4) is only an approximate equation for the displacement $u(t)$. In particular, Eq. (3) should be viewed as approximations for the spring force. In our

derivation we have also neglected the mass of the spring in comparison with the mass of the attached body.

The general solution of Eq. (4) is

$$u = A\cos\omega_0 t + B\sin\omega_0 t, \tag{5}$$

where $\omega_0^2 = k/m$.

The arbitrary constants $A$ and $B$ can be determined if initial conditions of the form

$$u(0) = u_0,\ u'(0) = v_0. \tag{6}$$

are given.

In discussing the solution of Eq. (4) it is convenient to rewrite Eq. (5) in the form

$$u = R\cos(\omega_0 t - \delta), \tag{7}$$

where

$$R = \sqrt{A^2 + B^2}\ ,\ \tan\delta = B/A.$$

In calculating $\delta$ care must be taken to choose the correct quadrant; this can be done by checking the signs of $\cos\delta$ and $\sin\delta$ in Eqs. (5).

The graph of Eq. (7), or the equivalent Eq. (5), for a typical set of initial conditions is shown in Figure 4.3. The graph is a displaced cosine wave that describes a periodic, or simple harmonic, motion of the mass. The **period** of the motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}} \tag{8}$$

The circular frequency $\omega 0 = \sqrt{\frac{m}{k}}$, measured in radians per unit time, is called the **natural frequency** of the vibration. The maximum displacement $R$ of the mass from equilibrium is the **amplitude** of the motion. The dimensionless parameter $\delta$ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.

Note that the motion described by Eq. (7) has a constant amplitude that does not diminish with time. This reflects the fact that in the absence of damping there is no way for the system to dissipate the energy imparted to it by the initial displacement and velocity. Further, for a given mass $m$ and spring constant $k$, the system always vibrates at the same frequency $\omega_0$, regardless of the initial conditions. However, the initial conditions do help to determine the amplitude of the motion. Finally, observe from Eq. (8) that $T$ increases as $m$ increases, so larger masses vibrate more slowly. On the other hand, $T$ decreases as $k$ increases, which means that stiffer springs cause the system to vibrate more rapidly.

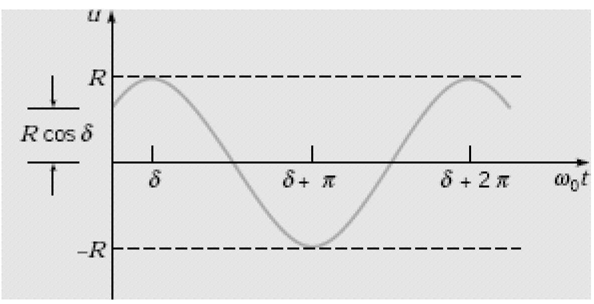



**FIG. 4.3** Simple harmonic motion; $u = R\cos(\omega_0 t - \delta)$.

**4.2 Damped Systems.** In this case, beside the two forces (**1. The weight** and **2. The spring force**) as above, we have to consider one more force, that is:
**3. The damping or resistive force** $F_d$ always acts in the direction opposite to the direction of motion of the mass. This force may arise from several sources: resistance from the air or other medium in which the mass moves, internal energy dissipation due to the extension or compression of the spring, friction between the mass and the guides (if any) that constrain its motion to one dimension, or a mechanical device (dashpot) that imparts a resistive force to the mass. In any case, we assume that the resistive force is proportional to the speed $|du/dt|$ of the mass; this is usually referred to as viscous damping. If $du/dt > 0$, then $u$ is increasing, so the mass is moving downward. Then $F_d$ is directed upward and is given by

$$F_d(t) = -\gamma u'(t), \tag{8}$$

where $\gamma$ is a positive constant of proportionality known as the damping constant. On the other hand, if $du/dt < 0$, then $u$ is decreasing, the mass is moving upward, and $F_d$ is directed downward. In this case, $Fd = \gamma |u'(t)|$; since $|u'(t)| = -u'(t)$, it follows that $F_d(t)$ is again given by Eq. (8). Thus, regardless of the direction of motion of the mass, the damping force is always expressed by Eq. (8).
The damping force may be rather complicated and the assumption that it is modelled adequately by Eq. (8) may be open to question. Some dashpots do behave as Eq. (8) states, and if the other sources of dissipation are small, it may be possible to neglect them altogether, or to adjust the damping constant $\gamma$ to approximate them. An important benefit of the assumption leading to Eq. (8) is that it leads to a linear (rather than a nonlinear) differential equation. In turn, this means that a thorough analysis of the system is straightforward, as we will show in this section and the next.
Taking account of these forces, we can now rewrite Newton's law as

$$mu''(t) = mg + F_S(t) + F_d(t) = mg - k[L + u(t)] - \gamma u'(t)$$

Since $mg - kL = 0$, it follows that the equation of motion of the mass is

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \tag{9}$$

where the constants $m$, $\gamma$ , and $k$ are positive. Note that Eq. (9) has the same form as Eq. (2).

We are especially interested in examining the effect of variations in the damping coefficient $\gamma$ for given values of the mass $m$ and spring constant $k$. The roots of the corresponding characteristic equation are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right).$$

Depending on the sign of $\gamma^2 - 4km$, the solution $u$ has one of the following forms:

$$\gamma^2 - 4km > 0, \quad u = Ae^{r_1 t} + Be^{r_2 t};$$
$$\gamma^2 - 4km = 0, \quad u = (A + Bt)e^{-\gamma t/2m};$$
$$\gamma^2 - 4km < 0, \quad u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \quad \mu = \frac{(4km - \gamma^2)^{1/2}}{2m} \tag{10}$$

Since $m$, $\gamma$ , and $k$ are positive, $\gamma^2 - 4km$ is always less than $\gamma^2$. Hence, if $\gamma^2 - 4km \geq 0$, then the values of $r_1$ and $r_2$ given by above formulae are *negative*. If $\gamma^2 - 4km < 0$, then the values of $r_1$ and $r_2$ are complex, but with *negative* real part. Thus, in all cases, the solution $u$ tends to zero as $t \to \infty$; this occurs regardless of the values of the arbitrary constants $A$ and $B$, that is, regardless of the initial conditions. This confirms our intuitive expectation, namely, that damping gradually dissipates the energy initially imparted to the system, and consequently the motion dies out with increasing time.
The most important case is the third one, which occurs when the damping is small.
If we let $A = R\cos\delta$ and $B = R\sin\delta$ in Eq. (10), then we obtain

$$u = Re^{-\gamma t/2m}\cos(\mu t - \delta).$$

The displacement $u$ lies between the curves $u = \pm Re^{-\gamma t/2m}$; hence it resembles a cosine wave whose amplitude decreases as $t$ increases. A typical example is sketched in Figure 3.8.5. The

motion is called a damped oscillation, or a damped vibration. The amplitude factor $R$ depends on $m$, $\gamma$ , $k$, and the initial conditions.
Although the motion is not periodic, the parameter $\mu$ determines the frequency with which the mass oscillates back and forth; consequently, $\mu$ is called the **quasi frequency**. By comparing $\mu$ with the frequency $\omega_0$ of undamped motion, we find that

$$\frac{\mu}{\omega_0} = \frac{(4km - \gamma^2)^{1/2}/2m}{\sqrt{k/m}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \cong 1 - \frac{\gamma^2}{8km}. \qquad (11)$$

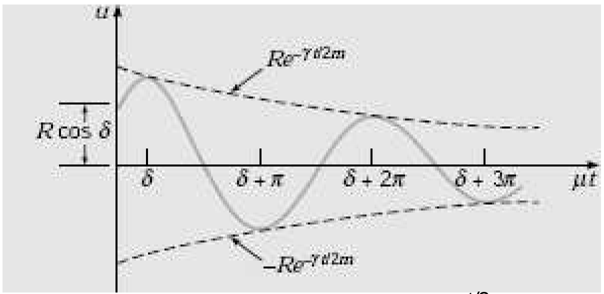



**FIG. 4.4** Damped oscillation; $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$.

The last approximation is valid when $\gamma^2/4km$ is small; we refer to this situation as "small damping." Thus, the effect of small damping is to reduce slightly the frequency of the oscillation. The quantity $Td = 2\pi/\mu$ is called the **quasi period**. It is the time between successive maxima or successive minima of the position of the mass, or between successive passages of the mass through its equilibrium position while going in the same direction. The relation between $T_d$ and $T$ is given by

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1 + \frac{\gamma^2}{8km}\right), \qquad (12)$$

where again the last approximation is valid when $\gamma^2/4km$ is small. Thus, small damping increases the quasi period.
Equations (11) and (12) reinforce the significance of the dimensionless ratio $\gamma^2/4km$.
It is not the magnitude of $\gamma$ alone that determines whether damping is large or small, but the magnitude of $\gamma^2$ compared to $4km$. When $\gamma^2/4km$ is small, then we can neglect the effect of damping in calculating the quasi frequency and quasi period of the motion. On the other hand, if we want to study the detailed motion of the mass for all time, then we can *never* neglect the damping force, no matter how small.

## 5. Nonhomogeneous Equations: Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \qquad (1)$$

where $p$, $q$, and $g$ are given (continuous) functions on the open interval $I$ . The equation

$$y'' + p(t)y' + q(t)y = 0, \qquad (2)$$

in which $g(t) = 0$ and $p$ and $q$ are the same as in Eq. (1), is called the homogeneous equation corresponding to Eq. (1). The following two results describe the structure of solutions of the nonhomogeneous equation (1) and provide a basis for constructing its general solution.

**5.1. Theorem.** If $Y_1$ and $Y_2$ are two solutions of the nonhomogeneous equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, $y_1$ and $y_2$ are a fundamental set of solutions of Eq. (2), then

$$Y_1(t) - Y_2(t) = c_1\, y_1(t) + c_2\, y_2(t), \qquad (3)$$

where $c_1$ and $c_2$ are certain constants.

**Proof.** To prove this result, note that $Y_1$ and $Y_2$ satisfy the equations (1), this means that

$$Y_1'' + p(t)\, Y_1' + q(t)Y_1 = g(t) \text{ and } Y_1'' + p(t)\, Y_1' + q(t)Y_1 = g(t) \quad (4)$$

Subtracting the second of these equations from the first, we have

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0. \quad (5)$$

Equation (5) states that $Y_1 - Y_2$ is a solution of Eq. (2). Finally, since all solutions of Eq. (2) can be expressed as linear combinations of a fundamental set of solutions by Theorem 2.9, it follows that the solution $Y_1 - Y_2$ can be so written. Hence Eq. (3) holds and the proof is complete.

**5.2 Theorem.** The general solution of the nonhomogeneous equation (1) can be written in the form

$$y = \varphi(t) = c_1\, y_1(t) + c_2\, y_2(t) + Y(t), \quad (6)$$

where $y_1$ and $y_2$ are a fundamental set of solutions of the corresponding homogeneous equation (2), $c_1$ and $c_2$ are arbitrary constants, and $Y$ is some specific solution of the nonhomogeneous equation (1).
The proof of Theorem 5.2 follows quickly from the preceding theorem. Note that Eq. (3) holds if we identify $Y_1$ with an arbitrary solution $\varphi$of Eq. (1) and $Y_2$ with the specific solution $Y$ . From Eq. (3) we thereby obtain

$$\varphi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t), \quad (7)$$

which is equivalent to Eq. (6). Since $\varphi$ is an arbitrary solution of Eq. (1), the expression on the right side of Eq. (7) includes all solutions of Eq. (1); thus it is the general solution of Eq. (1).

In somewhat different words, Theorem 5.2 states that to solve the nonhomogeneous equation (1), we must do three things:

**1.** Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation. This solution is frequently called the complementary solution and may be denoted by $y_c(t)$.

**2.** Find some single solution $Y(t)$ of the nonhomogeneous equation. Often this solution is referred to as a particular solution.

**3.** Add together the functions found in the two preceding steps.

We have already discussed how to find $y_c(t)$, at least when the homogeneous equation (2) has constant coefficients. Therefore, in the remainder of this section and in the next, we will focus on finding a particular solution $Y(t)$ of the nonhomogeneous equation (1). There are two methods that we wish to discuss. They are known as the method of undetermined coefficients and the method of variation of parameters, respectively. Each has some advantages and some possible shortcomings.

***5.3 Method of Undetermined Coefficients.*** The method of undetermined coefficients requires that we make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified. We then substitute the assumed expression into Eq. (1) and attempt to determine the coefficients so as to satisfy that equation. Then we have found a solution of the differential equation (1) and can use it for the particular solution $Y(t)$. The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of $Y(t)$. Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. For this reason, this method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sinus, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications. Precisely, we will consider the second-order linear differential equations with constant coefficients:

$$y'' + ay' + by = g(t) \qquad (8)$$

and with the following forms of the nonhomogeneous term g(t):

**FORM 1:** $g(t)=e^{\alpha t}P_n(t)$, where $P_n(t)$ is a polynomial of order n. For g(t) of this form we consider the following cases:

Case I: The constant $\alpha$ is not a root of the characteristic equation $k^2+ak+b = 0$. In this case, we choose $Y= e^{\alpha t}Q_n(t)$ with $Q_n(t)$ being a polynomial of degree n whose coefficients are found by substituting Y to Eq. (8).

Case II: The constant $\alpha$ is a single root of the characteristic equation $k^2+ak+b = 0$. In this case, we choose $Y= te^{\alpha t}Q_n(t)$ with $Q_n(t)$ being a polynomial of degree n whose coefficients are found by substituting Y to Eq. (8).

Case III: The constant $\alpha$ is the double root of the characteristic equation $k^2+ak+b = 0$. In this case, we choose $Y= t^2e^{\alpha t}Q_n(t)$ with $Q_n(t)$ being a polynomial of degree n whose coefficients are found by substituting Y to Eq. (8).

**EXAMPLES**

**1.** Consider $$y'' +3y' - 4y = t; \ (\alpha=0;\ n=1) \qquad (9)$$

The corresponding homogeneous equation is $y''+3y' -4y =0$ with the characteristic equation

$$k^2+3k-4=0 \Leftrightarrow k=1 \text{ or } -4.$$

Therefore, the general solution of the corresponding homogeneous equation is $c_1e^t+c_2e^{-4t}$. Since $\alpha=0$ is not a root of characteristic equation, we find a particular solution of Eq. (9) of the form $Y=At+B$; Substituting this form into (9) we obtain that $-4At+3A-4B=t$. Identifying the corresponding coefficients of t we have that $A=-1/4$ and $B=-3/16$. This yields a particular solution of Eq. (9) as $Y=-\frac{1}{4}t-\frac{3}{16}$ and hence, the general solution of (9):

$$y= c_1e^t+c_2e^{-4t}-\frac{1}{4}t-\frac{3}{16}$$

**2.** Consider $$y'' - y' = e^t(t+1); \ (\alpha=1;\ n=1) \qquad (10)$$

The corresponding homogeneous equation is $y'' - y'=0$ having the general solution as $c_1e^t+c_2e^{-t}$. Since $\alpha=1$ is a single root of the characteristic equation, we find a particular solution of Eq. (10) of the form $Y=te^t(At+B)$; Substituting this form into (10) we obtain that $e^t(2At+B+2A)=e^t(t+1)$ $\Rightarrow A=1/2$ and $B=0$. Therefore, the general solution of (10) is

$$y= c_1e^t+c_2e^{-t}+\frac{1}{2}t^2e^t$$

**3.** Consider $$y'' - 2y'+y = e^t; \ (\alpha=1;\ n=0) \qquad (11)$$

The corresponding homogeneous equation is $y'' - 2y'+y=0$ having the general solution as $(c_1+c_2t)e^t$. Since $\alpha=1$ is the double root of the characteristic equation, we find a particular solution of Eq. (11) of the form $Y=At^2e^t$; Substituting this form into (11) we obtain that $A=1/2$ .

Therefore, the general solution of (11) is $y= (c_1+c_2t)e^t+\frac{1}{2}t^2e^t$

**FORM 2:** $g(t)=P_m(t)\cos\beta t + Q_n(t)\sin\beta t$, where $P_m(t)$ and $Q_n(t)$ are known polynomials of order m and n, respectively. For g(t) of this form we consider following cases:

Case I: The constant $i\beta$ is not a root of the characteristic equation $k^2+ak+b = 0$. In this case, we choose $Y= R_\ell(t)\cos\beta t+S_\ell(t)\sin\beta t$ with $R_\ell(t)$ and $S_\ell(t)$ being polynomials of degree $\ell = \max\{m, n\}$ whose coefficients are found by substituting Y to Eq. (8).

Case II: The constant $i\beta$ is a root of the characteristic equation $k^2+ak+b = 0$. In this case, we choose $Y= t(R_\ell(t)\cos\beta t+S_\ell(t)\sin\beta t)$ with $R_\ell(t)$ and $S_\ell(t)$ being polynomials of degree $\ell$ whose coefficients are found by substituting Y to Eq. (8).

**EXAMPLE**

Consider

$$y'' + 4y = 3\cos 2t; \qquad (\beta=2, \ell= 0) \tag{12}$$

Firstly, solve the homogeneous equation

$$y''+ 4y = 0 \tag{13}$$

that corresponds to Eq. (12). The characteristic equation $k^2+4=0$ has two complex conjugate root $k = \pm 2i$. Therefore, the general solution of Eq. (13) is $c_1\cos 2t + c_2\sin 2t$.

Since the nonhomogeneous term is 3 cos 2*t*, and 2i is a root of the characteristic equation, we will find a particular of the form $Y(t) = At\cos 2t + Bt\sin 2t$. Then, upon calculating $Y'(t)$ and $Y''(t)$, substituting them into Eq. (12), and collecting terms, we find that

$$-4A\sin 2t + 4B\cos 2t = 3\cos 2t.$$

Therefore $A = 0$ and $B = \frac{3}{4}$, so a particular solution of Eq. (20) is $Y(t) = \frac{3}{4}t\sin 2t$. Hence,

The general solution of (12) is $y(t)= c_1\cos 2t + c_2\sin 2t+\frac{3}{4}t\sin 2t$.

**FORM 3:** $g(t) = e^{\alpha t}[P_m(t)\cos\beta t+Q_n(t)\sin\beta t]$; In this case, we use the substitution $y= e^{\alpha t}z$ with that the equation (8) becomes

$$e^{\alpha t}[z''+(2\alpha+a)z'+(\alpha^2+a\alpha+b)z] = e^{\alpha t}[P_m(t)\cos\beta t+Q_n(t)\sin\beta t]$$

dividing by $e^{\alpha t}$ on both sides we obtain that

$$z''+(2\alpha+a)z'+(\alpha^2+a\alpha+b)z = P_m(t)\cos\beta t+Q_n(t)\sin\beta t \tag{14}$$

This equation has the form 2 and hence can be solved for z. Returning to y by using the above formula of substitution we obtain the solution of (8).

**Note on the case of Form 3:** An alternative way to solve (8) in this case is to find a particular solution of (8) of the form $Y(t)= e^{\alpha t}[R_\ell(t)\cos\beta t+S_\ell(t)\sin\beta t]$ if $\alpha + i\beta$ is not a root of the characteristic equation $k^2+ak+b = 0$, or of the form $Y(t)= te^{\alpha t}[R_\ell(t)\cos\beta t+S_\ell(t)\sin\beta t]$ if $\alpha + i\beta$ is a root of the characteristic equation.

**EXAMPLE**

Solve the equation $$y''- 2y' +5y = 3e^t\cos 2t. \tag{15}$$

Substituting $y= e^tz$ to (15) we obtain: $z''+4z=3\cos 2t$.

This equation is precisely the Eq. (12) above and has the general solution as

$$z(t)= c_1\cos 2t + c_2\sin 2t+\frac{3}{4}t\sin 2t.$$

Then, the general solution of (15) is $y= e^tz=e^t(c_1\cos 2t + c_2\sin 2t+\frac{3}{4}t\sin 2t)$.

## 5.4 Superposition of solutions

Now suppose that $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$, and suppose that $Y_1$ and $Y_2$ are solutions of the equations

$$y'' + ay' + by = g_1(t) \tag{16}$$

and

$$y'' + ay' + by = g_2(t), \tag{17}$$

respectively. Then $Y_1 + Y_2$ is a solution of the equation

$$y'' + ay' + by = g(t). \tag{18}$$

To prove this statement, substitute $Y_1(t) + Y_2(t)$ for $y$ in Eq. (18) and make use of Eqs. (16) and (17). A similar conclusion holds if $g(t)$ is the sum of any finite number of terms.

The practical significance of this result is that for an equation whose nonhomogeneous function $g(t)$ can be expressed as a sum, one can consider instead several simpler equations and then add the results together. The following example is an illustration of this procedure.

**EXAMPLE**

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \tag{19}$$

By splitting up the right side of Eq. (19), we obtain the three equations

$$y'' - 3y' - 4y = 3e^{2t},$$

$$y'' - 3y' - 4y = 2\sin t,$$

and

$$y'' - 3y' - 4y = -8e^{t}\cos 2t.$$

Solutions of these three equations are $Y_1=-e^{2t}/2$, $Y_1=(3\cos t - 5\sin t)/17$, and $Y_3=e^{t}(10\cos 2t + 2\sin 2t)/13$, respectively.
Therefore a particular solution of Eq. (19) is their sum, namely,

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t.$$

## 6. Variation of Parameters

In this section we describe another method of finding a particular solution of a nonhomogeneous equation. This method, known as **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires that we evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties.

Again we consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{20}$$

and the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{21}$$

As a starting point, we assume that we know the general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{22}$$

of the corresponding homogeneous equation (21).

This is a major assumption because so far we have shown how to solve Eq. (21) only if it has constant coefficients. If Eq. (21) has coefficients that depend on $t$, then usually the methods described in previous sections must be used to obtain $y(t)$.

The crucial idea is to replace the constants $c_1$ and $c_2$ in Eq. (22) by functions $u_1(t)$ and $u_2(t)$, respectively; this gives

$$y = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{23}$$

Then we try to determine $u_1(t)$ and $u_2(t)$ so that the expression in Eq. (23) is a solution of the nonhomogeneous equation (20) .
Thus we differentiate Eq. (23), obtaining

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t). \tag{24}$$

To determine $u_1$ and $u_2$ we need to substitute for $y$ from Eq. (23) in Eq. (20). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of $u_1$, $u_2$, and their first two derivatives.
Since there is only one equation and two unknown functions, we can expect that there are many possible choices of $u_1$ and $u_2$ that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions $u_1$ and $u_2$. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.

We now set the terms involving $u_1'(t)$ and $u_2'(t)$ in Eq. (24) equal to zero; that is, we require that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{25}$$

Then, we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{26}$$

Further, by differentiating again, we obtain

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t). \tag{27}$$

Now we substitute for $y$, $y'$, and $y''$ in Eq. (20) from Eqs. (23), (26), and (27), respectively. After rearranging the terms in the resulting equation we find that

$$u_1(t)[y_1''(t) + p(t)y_1(t) + q(t)y_1(t)] + u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] + \\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{28}$$

Each of the expressions in square brackets in Eq. (28) is zero because both $y_1$ and $y_2$ are solutions of the homogeneous equation (21). Therefore Eq. (28) reduces to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{29}$$

Equations (25) and (29) form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. The coefficient matrix is $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ whose determinant is Wronskian $W(y_1,y_2)\neq 0$ since $y_1$, $y_2$ are linearly independent. By solving the system (25), (29) we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)}; \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)} \tag{30}$$

By integrating Eqs. (30) we find the desired functions $u_1(t)$ and $u_2(t)$, namely,

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)}dt; \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)}dt. \tag{31}$$

Therefore, we obtain a particular solution of (20) given by formula (23) with $u_1(t)$ and $u_2(t)$ being determined by (31).

We state the result as a theorem.
**Theorem 6.1.** If the functions $p$, $q$, and $g$ are continuous on an open interval $I$, and if the functions $y_1$ and $y_2$ are linearly independent solutions of the homogeneous equation (21) corresponding to the nonhomogeneous equation (20),

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (20) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1,y_2)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)}dt. \tag{32}$$

and the general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t). \quad (33)$$

By examining the expression (32) and reviewing the process by which we derived it, we can see that there may be two major difficulties in using the method of variation of parameters. As we have mentioned earlier, one is the determination of $y_1(t)$ and $y_2(t)$, a fundamental set of solutions of the homogeneous equation (21), when the coefficients in that equation are not constants. The other possible difficulty is in the evaluation of the integrals appearing in Eq. (32). This depends entirely on the nature of the functions $y_1$, $y_2$, and $g$. In using Eq. (32), be sure that the differential equation is exactly in the form (20); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that Eq. (32) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions.

**EXAMPLE**

Find a particular solution of

$$y'' + 4y = 3 \csc t. \quad (33)$$

Observe that this problem does not fall within the scope of the method of undetermined coefficients because the nonhomogeneous term $g(t) = 3 \csc t = 1/\sin t$ involves a quotient (rather than a sum or a product) of $\sin t$ and $\cos t$. The homogeneous equation corresponding to Eq. (33) is

$$y'' + 4y = 0,$$

having the general solution as

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Replacing the constants $c_1$ and $c_2$ in this equation by functions $u_1(t)$ and $u_2(t)$, respectively, and then to determine these functions so that the resulting expression

$$y = u_1(t) \cos 2t + u_2(t) \sin 2t \quad (34)$$

is a solution of the nonhomogeneous equation.

With the additional requirement

$$u_1'(t) \cos 2t + u_2'(t) \sin 2t = 0, \quad (35)$$

and substituting y(t) from (34) into (33) we obtain that $u_1$ and $u_2$ must satisfy

$$-2u_1'(t) \sin 2t + 2u_2'(t) \cos 2t = 3 \csc t. \quad (36)$$

Solving the system of linear equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$ we find

$$u_1'(t) = -3 \csc t \sin 2t = -3 \cos t.$$

$$u_2'(t) = \frac{3}{2} \csc t - 3 \sin t.$$

Having obtained $u_1'(t)$ and $u_2'(t)$, the next step is to integrate so as to obtain $u_1(t)$ and $u_2(t)$. The result is

$$u_1(t) = -3 \sin t + c1$$

and

$$u_2(t) = \frac{3}{2} \ln|\csc t - \cot t| + 3 \cos t + c2.$$

Finally, on substituting these expressions in Eq. (34), we obtain the general solution of (33) as

$$y = -3 \sin t + \frac{3}{2} \ln|\csc t - \cot t| \sin 2t + c1 \cos 2t + c2 \sin 2t.$$

## 7. Modelling: Forced Oscillation

We have already known that the equation governed the mechanical system of free oscillation (Mass-spring problem) is

$$mu'' + \gamma u' + ku = 0,$$

where the term mu'' represents the force of inertia, $\gamma u'$-the damping force, and $ku$-spring force.
Now, forced motions are obtained if we let an external force g(t) act on the body. To get the model, we simply have to add our new force g(t) to these forces to obtain

$$mu'' + \gamma u' + ku = g(t).$$

Then, g(t) is called the input or driving force, and the corresponding solutions are called an output or a response of the system to the driving force.

Of particular interest are periodic input, say g(t)=$F_0 \cos \omega t$ with $\omega > 0$. Then the equation of motion is

$$mu'' + \gamma u' + ku = F_0 \cos \omega t. \qquad (1)$$

***Forced Vibrations without Damping:*** First suppose that there is no damping ($\gamma=0$); then Eq. (1) reduces to

$$mu'' + ku = F_0 \cos \omega t. \qquad (2)$$

If $\omega_0 = \sqrt{k/m} \neq \omega$ then the general solution of Eq. (2) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \qquad (3)$$

The constants $c_1$ and $c_2$ are determined by the initial conditions. The resulting motion is, in general, the sum of two periodic motions of different frequencies ( $\omega_0$ and $\omega$) and amplitudes. There are two particularly interesting cases.

**Beats.** Suppose that the mass is initially at rest, so that $u(0) = 0$ and $u''(0) = 0$. Then it turns out that the constants $c_1$ and $c_2$ in Eq. (3) are given by

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \qquad c_2 = 0, \qquad (4)$$

and the solution of Eq. (2) is

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \qquad (5)$$

This is the sum of two periodic functions of different periods but the same amplitude. Making use of the trigonometric identities for $\cos(A \pm B)$ with $A = (\omega_0 + \omega)t/2$ and $B = (\omega_0 - \omega)t/2$, we can write Eq. (5) in the form

$$u = \left[ \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \right] \sin \frac{(\omega_0 + \omega)t}{2}. \qquad (6)$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega$ is much greater than $|\omega_0 - \omega|$. Consequently, $\sin(\omega_0 + \omega)t/2$ is a rapidly oscillating function compared to $\sin(\omega_0 - \omega)t/2$. Thus the motion is a rapid oscillation with frequency $(\omega_0 + \omega)/2$, but with a slowly varying sinusoidal amplitude

$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}.$$

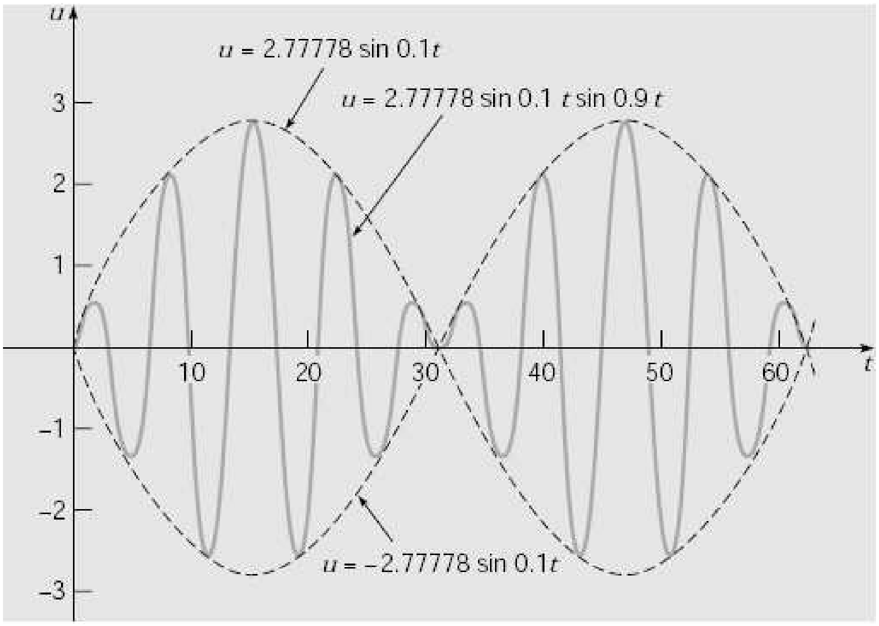



**Fig. 6.1:** A beat; solution of $u'' + u = 0.5\cos 0.8t$, $u(0) = 0$, $u'(0) = 0$;

$u = 2.77778 \sin 0.1t \sin 0.9t$.

This type of motion, possessing a periodic variation of amplitude, exhibits what is called a **beat**. Such a phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are sounded simultaneously. In this case the periodic variation of amplitude is quite apparent to the unaided ear. In electronics the variation of the amplitude with time is called **amplitude modulation**. The graph of *u* as given by Eq. (6) in a typical case is shown in the Figure 6.1.

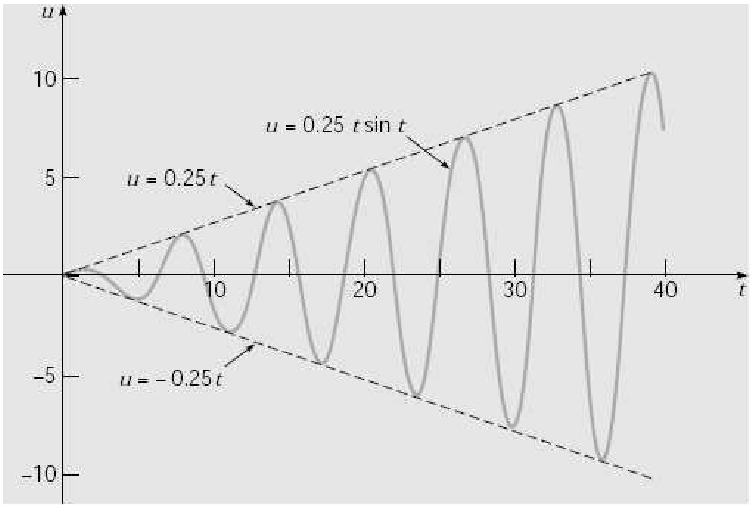
**Resonance.** As a second example, consider the case $\omega = \omega_0$; that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $F_0 \cos \omega t$ is a solution of the homogeneous equation. In this case the solution of Eq. (2) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \qquad (7)$$

Because of the term $t \sin \omega_0 t$, the solution (7) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of *c*1 and *c*2; see Figure 3.9.2 for a typical example. Of course, in reality unbounded oscillations do not occur. As soon as *u* becomes large, the mathematical model on which Eq. (1) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that *u* be small. If damping is included in the model, the predicted motion remains bounded; however, the response to the input function $F_0 \cos \omega t$ may be quite large if the damping is small and $\omega$ is close to $\omega_0$. This phenomenon is known as **resonance**.

Resonance can be either good or bad depending on the circumstances. It must be taken very seriously in the design of structures, such as buildings or bridges, where it can produce instabilities possibly leading to the catastrophic failure of the structure. For example, soldiers traditionally break step when crossing a bridge to eliminate the *periodic* force of their marching that could resonate with a natural frequency of the bridge. Another example occurred in the design of the high-pressure fuel turbopump for the space shuttle main engine. The turbopump was unstable and could not be operated over 20,000 rpm as compared to the design speed of 39,000 rpm. This difficulty led to a shutdown of the space shuttle program for

6 months at an estimated cost of \$500,000/day. On the other hand, resonance can be put to good use in the design of instruments, such as seismographs, intended to detect weak periodic incoming signals.




***Forced Vibrations with Damping.*** The motion of the spring–mass system with damping ($\gamma \neq 0$) and the forcing function $F0 \cos \omega t$ can be determined in a straightforward manner, although the computations are rather lengthy. The solution of Eq. (1) is

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t} + R\cos(\omega t - \delta), \quad r_1 \neq r_2 \tag{8}$$

where

$$R = \frac{F_0}{\Delta}, \qquad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \qquad \sin\delta = \frac{\gamma\omega}{\Delta}, \tag{9}$$

and

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{10}$$

In Eq. (8), $r_1$ and $r_2$ are the roots of the characteristic equation associated with Eq. (1); they may be either real and negative or complex conjugates with negative real part. In either case, both $\exp(r_1 t)$ and $\exp(r_2 t)$ approach zero as $t \to \infty$. Hence, as $t \to \infty$,

$$u \to U(t) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\cos(\omega t - \delta). \tag{11}$$

For this reason $u_c(t) = c_1 exp(r_1 t) + c_2 exp(r_2 t)$ is called the **transient solution**; $U(t)$, which represents a steady oscillation with the same frequency as the external force, is called the **steady-state solution** or the **forced response**. The transient solution enables us to satisfy what ever initial conditions are imposed; with increasing time the energy put into the system by the initial displacement and velocity is dissipated through the damping force, and the motion then becomes the response of the system to the external force. Without damping, the effect of the initial conditions would persist for all time.

It is interesting to investigate how the amplitude $R$ of the steady-state oscillation depends on the frequency $\omega$ of the external force. For low-frequency excitation, that is, as $\omega \to 0$, it follows from Eqs. (9) and (10) that $R \to F_0/k$. At the other extreme, for very high-frequency excitation, Eqs. (9) and (10) imply that $R \to 0$ as $\omega \to \infty$. At an intermediate value of $\omega$ the amplitude may have a maximum. To find this maximum point, we can differentiate $R$ with respect to $\omega$ and set the result equal to zero. In this way we find that the maximum amplitude occurs when $\omega = \omega_{max}$, where

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2km}\right). \tag{12}$$

Note that $\omega_{max} < \omega_0$ and that $\omega_{max}$ is close to $\omega_0$ when $\gamma$ is small. The maximum value of $R$ is

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right), \tag{13}$$

where the last expression is an approximation for small $\gamma$ . If $\gamma^2/2km > 1$, then $\omega_{max}$ as given by Eq. (12) is pure imaginary; in this case the maximum value of $R$ occurs for $\omega = 0$ and $R$ is a monotone decreasing function of $\omega$. For small $\gamma$ it follows from Eq. (13) that $R_{max} \cong F_0/\gamma\omega_0$. Thus, for small $\gamma$, the maximum response is much larger than the amplitude $F_0$ of the external force, and the smaller the value of $\gamma$ , the larger the ratio $R_{max}/F_0$. Figure 6.3 contains some representative graphs of $R_k/F_0$ versus $\omega/\omega_0$ for several values of $\gamma$ .

The phase angle $\delta$ also depends in an interesting way on $\omega$. For $\omega$ near zero, it follows from Eqs. (9) and (10) that $\cos\delta \cong 1$ and $\sin\delta \cong 0$. Thus $\delta \cong 0$, and the response is nearly in phase with the excitation, meaning that they rise and fall together, and in particular, assume their respective maxima nearly together and their respective minima nearly together. For $\omega = \omega_0$, we find that $\cos\delta = 0$ and $\sin\delta = 1$, so $\delta = \pi/2$. In this case the response lags behind the excitation by $\pi/2$; that is, the peaks of the response occur $\pi/2$ later than the peaks of the excitation, and similarly for the valleys. Finally, for $\omega$ very large, we have $\cos\delta \cong -1$ and $\sin\delta \cong 0$. Thus $\delta \cong \pi$, so that the response is nearly out of phase with the excitation; this means that the response is minimum when the excitation is maximum, and vice versa.

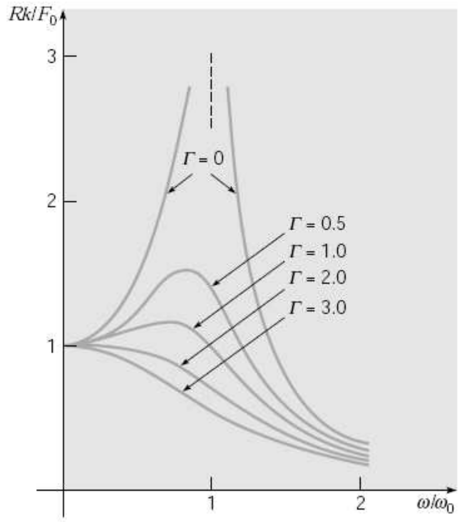



**FIGURE 6.3:** Forced vibration with damping: amplitude of steady-state response versus frequency of driving force;
$\Gamma = \gamma^2/m^2\omega_0^2$.

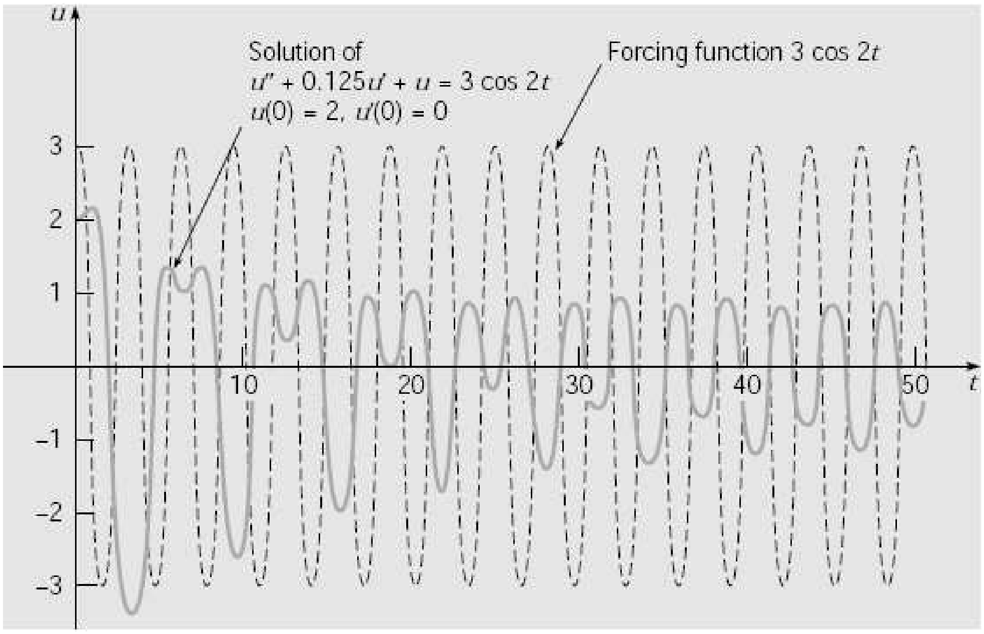



**FIGURE 6.4** A forced vibration with damping; solution of $u'' + 0.125u' + u = 3\cos 2t$, $u(0) = 2$, $u'(0) = 0$.

In Figure 6.4 we show the graph of the solution of the initial value problem

$$u'' + 0.125u' + u = 3\cos 2t, \quad u(0) = 2,\ u'(0) = 0.$$

The graph of the forcing function is also shown for comparison. Observe that the initial transient motion decays as $t$ increases, that the amplitude of the steady forced response is approximately 1, and that the phase difference between the excitation and response is approximately $\pi$.

More precisely, we find that $\Delta = \sqrt{145}/4 \approx 3.0104$, so $R = F_0/\Delta \approx 0.9965$. Furthermore, $\cos\delta = -3/\Delta \approx -0.9965$ and $\sin\delta = 1/4\Delta \approx 0.08305$, so that $\delta \approx 3.0585$. Thus the calculated values of $R$ and $\delta$ are close to the values estimated from the graph.

## 8. Power Series Solutions

**8.1. Definition:** The function f(x) is called real analytic at a point $x_0$ if it coincides with its Taylor's series of some neighbourhood $(x_0-R, x_0+R)$ of $x_0$, i.e., $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ for all $x \in (x_0-R, x_0+R)$.

The positive number R normally coincides with the radius of convergence of the Taylor's series.

**Examples:** $e^x$, $\sin x$, $\cos x$ are real analytic functions at any point $x_0 \in \mathbf{R}$.

The following theorem connects the real analyticity of coefficients of second order differential equations with their solutions**.**

**8.2. Theorem:** Consider equation

$$h(t)y'' + p(t)y' + q(t)y = r(t) \tag{8.1}$$

If h(t), p(t), q(t), and r(t) are analytic at $t=t_0$ with radius convergence R>0, then every solutions of (8.1) is also analytic at $t=t_0$ and can be represented by a power series in powers of $(t-t_0)$ with the same radius of convergence R.

Using Theorem 8.2 we have the following algorithm to find power series solutions of Equation (8.1):

***Step 1:*** Represent h(t), p(t), q(t) and r(t) by power series in powers of t (or of $(t-t_0)$ if solutions in powers of $(t-t_0)$ are wanted). Often, h(t), p(t), q(t), r(t) are polynomials, then nothing needs to be done in this step.

***Step 2:*** Write

$$y=\sum_{m=0}^{\infty} a_m t^m \text{ (or } y=\sum_{m=0}^{\infty} a_m (t-t_0)^m \text{ )}, \qquad (8.2)$$

then compute y' and y''.

***Step 3:*** Substitute y, y' and y'' obtained from Step 2 into (8.1). Then, collect the like powers of t and equate the sum of the coefficients of each occurring power of t to zero, starting from the constant terms, then the terms containing t, the terms containing $t^2$, …etc. This gives the relations from which we can determine the unknown coefficients in (8.2) successively.

**Example:** Consider $(1-t^2)y''-2ty'+2y=0.$ (8.4)
In this example, $h(t)=(1-t^2)$, $p(t)=-2t$, $q(t)=2$, $r(t)=0$ are already polynomials. We now write $y=\sum_{m=0}^{\infty} a_m t^m$, then compute $y'=\sum_{m=1}^{\infty} m a_m t^{m-1}$, and $y''=\sum_{m=2}^{\infty} m(m-1)a_m t^{m-2}$.
Next, we substitute y, y', y'' into Equation (8.4) to obtain

$$(1-t^2)\sum_{m=2}^{\infty} m(m-1)a_m t^{m-2} -2t\sum_{m=1}^{\infty} m a_m t^{m-1} +2\sum_{m=0}^{\infty} a_m t^m =0.$$

This is equivalent to

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=2}^{\infty} n(n-1)a_n t^n - \sum_{n=1}^{\infty} 2n a_n t^n +2\sum_{n=0}^{\infty} a_n t^n =0.$$

Collecting the like powers of t we have that:

$$2(a_0+a_2)+6a_3t+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+n(n-1)a_n\right]t^n = 0.$$

Equating the coefficients of each occurring power of t to zero, we obtain that

$a_0+a_2=0$; $a_3=0$; $a_{n+2}=-\dfrac{n(n-1)a_n}{(n+2)(n+1)}$ for $n\geq 2$. Therefore, by induction, we derive

$a_{2k+1}=0$ and $a_{2k}=\dfrac{(-1)^k a_0}{k(2k-1)}$ for all $k\geq 1$.

Hence, we obtain the general solution of (8.4) as

$$y = a_1 t+\sum_{k=1}^{\infty}\frac{(-1)^k a_0}{k(2k-1)}t^k \text{ for arbitrary } a_0 \text{ and } a_1.$$

# Problems

Consider the homogeneous equation $y'' + p(t)y' + q(t)y = 0$. (1)
with the coefficients being continuous on the interval *I*.

1. Show that two solutions of (1) on *I* that are zero at the same point in *I* cannot be linearly independent on *I*.

2. Show that two solutions of (1) on *I* that have maxima or minima at the same point in *I* cannot be linearly independent on *I*.

3. Suppose that $y_1$ and $y_2$ are two linearly independent solutions of (1) on *I*. Show that $z_1=a_{11}y_1+a_{12}y_2$ and $z_2=a_{21}y_1+a_{22}y_2$ (for some constants $a_{jk}$) form a basis of the solutions of (1) if and only if the determinant of the coefficients $a_{jk}$ is not zero.

4. Show that the equation $t^2y'' -4ty' + 6y=0$ *has* $y_1=t^2$ and $y_2=t^3$ as a basis of the solutions for all *t*. Show that $W(t^2, t^3)=0$ at t=0. Does this contradicts with Theorem 2.7 ?

**Reduction of order:** In each of Problems 5 through 9 show that the given function $y_1$ is a solution of the given equation. Using the method of reduction of the order, find $y_2$ such that $y_1$, $y_2$ form a basis. ***Caution!*** First write the equation in the standard form if you want to use the formula (9) in Section 2.11.

5. $(t+1)^2y''- 2(t+1)y' + 2y=0$, $y_1 = t +1$.
6. $(t-1)y''- 2ty' + y=0$, $y_1 = e^t$.
7. $(t-1)^2y''- 4(1-t)y' + 2y=0$, $y_1 = 1/(1-t)$.
8. $t^2y''+ ty' + (t^2-\frac{1}{4})y=0$, $y_1 = t^{-1/2}\cos t$.
9. $ty''+ 2y' + ty=0$, $y_1 = t^{-1}\sin t$.

In each of Problems 10 through 16 find the general solution of the given differential equation.

10. $y'' + 2y' - 3y = 0$
11. $y'' + 3y' + 2y = 0$
12. $6y'' - y' - y = 0$
13. $2y'' - 3y' + y = 0$
14. $y'' + 5y' = 0$
15. $4y'' - 9y = 0$
16. $y'' - 9y' + 9y = 0$

In each of Problems 17 through 24 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as *t* increases.

17. $y'' + y' - 2y = 0,\quad y(0) = 1,\quad y'(0) = 1$
18. $y'' + 4y' + 3y = 0,\quad y(0) = 2,\quad y'(0) = -1$
19. $6y'' - 5y' + y = 0,\quad y(0) = 4,\quad y'(0) = 0$
20. $y'' + 3y' = 0,\quad y(0) = -2,\quad y'(0) = 3$
21. $y'' + 5y' + 3y = 0,\quad y(0) = 1,\quad y'(0) = 0$
22. $2y'' + y' - 4y = 0,\quad y(0) = 0,\quad y'(0) = 1$
23. $y'' + 8y' - 9y = 0,\quad y(1) = 1,\quad y'(1) = 0$
24. $4y'' - y = 0,\quad y(-2) = 1,\quad y'(-2) = -1$

25. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.

26. Find a differential equation whose general solution is $y = c_1e^{-t/2} + c_2e^{-2t}$.

27. Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \frac{5}{4}, \quad y'(0) = -\frac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

28. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

29. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find $\alpha$ so that the solution approaches zero as $t \to \infty$.

30. Solve the initial value problem $4y'' - y = 0$, $y(0) = 2$, $y'(0) = \beta$. Then find $\beta$ so that the solution approaches zero as $t \to \infty$.

In each of Problems 31 through 34 find the Wronskian of two solutions of the given differential equation without solving the equation.

31. $t^2y'' - t(t+2)y' + (t+2)y = 0$

32. $(\cos t)y'' + (\sin t)y' - ty = 0$

33. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation

34. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, Legendre's equation

In each of Problems 35 through 44 find the general solution of the given differential equation.

35. $y'' - 2y' + 2y = 0$

36. $y'' + 2y' - 8y = 0$

37. $y'' + 6y' + 13y = 0$

38. $y'' + 2y' + 1.25y = 0$

39. $y'' + y' + 1.25y = 0$

40. $y'' - 2y' + 6y = 0$

41. $y'' + 2y' + 2y = 0$

42. $4y'' + 9y = 0$

43. $9y'' + 9y' - 4y = 0$

44. $y'' + 4y' + 6.25y = 0$

In each of Problems 45 through 47 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing *t*.

45. $y'' + 4y = 0$, $\quad y(0) = 0$, $\quad y'(0) = 1$

46. $y'' + 4y' + 5y = 0$, $\quad y(0) = 1$, $\quad y'(0) = 0$

47. $y'' - 2y' + 5y = 0$, $\quad y(\pi/2) = 0$, $\quad y'(\pi/2) = 2$

48. Consider the initial value problem $3u'' - u' + 2u = 0$, $u(0) = 2$, $u'(0) = 0$.
(a) Find the solution $u(t)$ of this problem.
(b) Find the first time at which $|u(t)| = 10$.

49. Consider the initial value problem $5u'' + 2u' + 7u = 0$, $u(0) = 2$, $u'(0) = 1$.
(a) Find the solution $u(t)$ of this problem.
(b) Find the smallest $T$ such that $|u(t)| \le 0.1$ for all $t > T$.

50. **Euler Equations.** An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0, \; t > 0,$$

where $\alpha$ and $\beta$ are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transforms an Euler equation into an equation with constant coefficients. Then, using this substitution to solve the following equations

a) $t^2 y'' - 3ty' + 4y = 0$, $t > 0$. b) $t^2 y'' + 2ty' + 0.25y = 0$, $t > 0$.

In each of Problems 51 through 60 find the general solution of the given differential equation.

51. $y'' - 2y' + y = 0$
52. $4y'' - 4y' - 3y = 0$
53. $y'' - 2y' + 10y = 0$
54. $4y'' + 17y' + 4y = 0$
55. $25y'' - 20y' + 4y = 0$
56. $9y'' + 6y' + y = 0$
57. $4y'' + 12y' + 9y = 0$
58. $y'' - 6y' + 9y = 0$
59. $16y'' + 24y' + 9y = 0$
60. $2y'' + 2y' + y = 0$

In each of Problems 61 through 64 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing *t*.

61. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
62. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
63. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$
64. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

65. If $a$, $b$, and $c$ are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

66. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 65 is no longer true, but that all solutions are bounded as $t \to \infty$.
(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 65 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y_1$.

In each of Problems 67 through 78 find the general solution of the given differential equation.

67. $y'' - 2y' - 3y = 3e^{2t}$
68. $y'' - 2y' - 3y = -3te^{-t}$
69. $y'' + 9y = t^2 e^{3t} + 6$
70. $2y'' + 3y' + y = t^2 + 3\sin t$
71. $u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2$
72. $y'' + y' + 4y = 2\sinh t$    *Hint*: $\sinh t = (e^t - e^{-t})/2$
73. $y'' - y' - 2y = \cosh 2t$    *Hint*: $\cosh t = (e^t + e^{-t})/2$
74. $y'' + 2y' + 5y = 3\sin 2t$
75. $y'' + 2y' = 3 + 4\sin 2t$
76. $y'' + 2y' + y = 2e^{-t}$
77. $y'' + y = 3\sin 2t + t\cos 2t$
78. $u'' + \omega_0^2 u = \cos \omega_0 t$

In each of Problems 74 through 80:
(a) Determine a suitable form for *Y(t)* if the method of undetermined coefficients is to be used.
(b) Find a particular solution of the given equation.

74. $y'' + y = t(1 + \sin t)$
75. $y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4)\sin t$
76. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t$
77. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$

78. $y'' + 4y = t^2 \sin 2t + (6t + 7)\cos 2t$

79. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$

80. $y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$

In each of Problems 81 through 88 find the general solution of the given differential equation. In Problems 11 and 12, *g(t)* is an arbitrary continuous function.

81. $y'' + y = \tan t, \quad 0 < t < \pi/2$

82. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$

83. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$

84. $y'' - 5y' + 6y = g(t)$

85. $y'' + 9y = 9\sec^2 3t, \quad 0 < t < \pi/6$

86. $y'' + 4y = 3\csc 2t, \quad 0 < t < \pi/2$

87. $y'' - 2y' + y = e^t/(1 + t^2)$

88. $y'' + 4y = g(t)$

89. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position *u* of the mass at any time *t*. Plot *u* versus *t*. Find the frequency, period, and amplitude of the motion.

90. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping, determine the position *u* of the mass at any time *t*. When does the mass first return to its equilibrium position?

91. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/sec, and if there is no damping, find the position *u* of the mass at any time *t*. Determine the frequency, period, amplitude, and phase of the motion.

92. A series circuit has a capacitor of $0.25 \times 10^{-6}$ farad and an inductor of 1 henry. If the initial charge on the capacitor is $10^{-6}$ coulomb and there is no initial current, find the charge *Q* on the capacitor at any time *t*.

93. A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne-sec/cm. If the mass is pulled down an additional 2 cm and then released, find its position *u* at any time *t*. Plot *u* versus *t*. Determine the quasi frequency and the quasi period. Determine the ratio of the quasi period to the period of the corresponding undamped motion. Also find the time $\tau$ such that $|u(t)| < 0.05$ cm for all $t > \tau$.

94. A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of 2 cos 3*t* lb, formulate the initial value problem describing the motion of the mass.
(a) Find the solution.
(b) Plot the graph of the solution.
(c) If the given external force is replaced by a force 4 sin $\omega t$ of frequency $\omega$, find the value of $\omega$ for which resonance occurs.

95. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of 10 sin(*t*/2) N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.

(a) Find the solution of the initial value problem.
(b) Identify the transient and steady-state parts of the solution.
(c) Plot the graph of the steady-state solution.
(d) If the given external force is replaced by a force 2 cos $\omega t$ of frequency $\omega$, find the value of $\omega$ for which the amplitude of the forced response is maximum.

96. If an undamped spring–mass system with a mass that weighs 6 lb and a spring constant 1 lb/in. is suddenly set in motion at $t = 0$ by an external force of 4 cos 7$t$ lb, determine the position of the mass at any time and draw a graph of the displacement versus $t$.

97. A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of 8 sin 8$t$ lb. If the mass is pulled down 3 in. and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

98. Find the power series solutions (in powers of x) of the following equations:

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
3. $xy'' + y = 0$
4. $xy'' + y' - y = 0$
5. $3x^2y'' + 2xy' + x^2y = 0$
6. $x^2y'' + xy' + (x - 2)y = 0$
7. $xy'' + (1 - x)y' - y = 0$
8. $2x^2y'' + 3xy' + (2x^2 - 1)y = 0$
9. $x^2y'' - x(x + 3)y' + (x + 3)y = 0$
10. $x^2y'' + (x^2 + \frac{1}{4})y = 0$

# CHAPTER 6: LAPLACE TRANSFORM

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in previous chapters are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those arising in engineering applications.

## 1. Definition and Domain

**1.1. Definition:** Let $f(t)$ be a given function defined on $\mathbf{R}_+=[0,\infty)$ and be piecewise continuous on every finite interval. If the following integral exists (i.e. it has a finite value)

$$\int_0^\infty e^{-st} f(t)dt$$

for s in some domain D, then we define a function *F(s)* by

$$F(s) = \int_0^\infty e^{-st} f(t)dt \text{ for } s\in D, \qquad (1)$$

and call it the *Laplace transform* of the function *f(t).* In this case, the function *f(t)* is called the *original function.* The operator $\mathscr{L}$, which assigns each original function *f(t)* to its Laplace transform *F(s),* is called the Laplace transform. Therefore, the Laplace transform *F* of *f* is $F=\mathscr{L}(f)$. Note that, sometimes, especially in physical problems, we use the notation

$f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s)$ to indicate the fact that $F=\mathscr{L}(f)$.

**Example:** 1) $f(t)=1$ for all $t\geq 0$, then $F(s)=\mathscr{L}(f)(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}$ for s>0. So, in other notation we can write: $1 \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s}$.

Here, the domain of definition of *F(s)* is $(0,\infty)$

2) $f(t)=e^{at}$ for all $t\geq 0$ (a-constant), then the Laplace transform of *f* is $F(s)=\int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}$ for s>a. Or, $e^{at} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s-a}$

From the above examples we remark that the domain of definition of the Laplace transform contains a half infinite interval. This remark is true in more general situations as we have the following theorem

**1.2 Theorem:** Let f(t) be a function that is defined and piecewise continuous on every finite intervals on the range $t\geq 0$, and satisfies

$$|f(t)|\leq Me^{\gamma t} \quad \forall\ t\geq 0 \qquad (2)$$

for some constants M and $\gamma$. Then, the Laplace transform of f(t) exists for all $s>\gamma$. (In this case, f(t) is called exponentially bounded; and $\gamma$ is called the growth bound of f(t).)

PROOF: The integral $\int_0^\infty e^{-st} f(t)dt$ exists if the integral $\int_0^\infty |e^{-st} f(t)|\,dt$ does. We now see that $|e^{-st} f(t)| \le Me^{(\gamma-s)t}$, and also the integral $\int_0^\infty Me^{(\gamma-s)t}dt$ is convergent for s>γ. Therefore, for s>γ $\int_0^\infty |e^{-st} f(t)|\,dt$ exists and hence so does $\int_0^\infty e^{-st} f(t)dt$.

**1.3 Theorem:** Let *f(t)* and *g(t)* be functions that are defined and piecewise continuous on the range $t\ge 0$. Suppose that they are exponentially bounded with the growth bounds $\gamma_1, \gamma_2$, respectively. Then, if $\mathscr{L}(f)(s) = \mathscr{L}(g)(s)$ for all $s>\max\{\gamma_1, \gamma_2\}$, we have that *f(t) = g(t)* at every continuous points of *f* and *g*.
Therefore, if two continuous functions have the same Laplace transforms, they are completely identical.

This means that, omitted the discontinuous points of the functions, we have that the relation between an original function and its Laplace transform is one-to-one.

Thus, the original function f(t) in (1) is called the inverse Laplace transform of F and is denoted by $\mathscr{L}^{-1}$(F). It is proved that, under some conditions, the original function f(t) can be reconstructed from F(s) by the formula

$$f(t) = \frac{1}{2\pi i}\int_{-\infty+i\sigma}^{\infty+i\sigma} F(s)e^{st}ds \quad \text{for some large enough } \sigma.$$

We note that the original functions are denoted by lowercase letters, and their Laplace transforms--by the same letters in capitals, e.g., F= $\mathscr{L}(f)$, $G= \mathscr{L}(g)$, etc.

## 2. Properties

**2.1. Linearity:** For all piecewise continuous functions *f, g*, and constants a, b we have

$$\mathscr{L}(af+bg)= a\mathscr{L}(f)+b\mathscr{L}(g).$$

Physically, one writes: $af(t)+bg(t) \overset{\mathscr{L}}{\longleftrightarrow} aF(s)+bG(s)$.

**Examples:** 1) Let f(t)=cosh(at)=($e^{at}+ e^{-at}$)/2. Find F= $\mathscr{L}$(f).

We already have $e^{at} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s-a}$ and $e^{-at} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}$. Therefore,

$$\frac{1}{2}(e^{at}+ e^{-at}) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right).$$

Hence, F(s)= $\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}$.

2) Let F(s)= $\frac{1}{(s-a)(s-b)}$; a ≠ b. Find f=$\mathscr{L}^{-1}$(F).

We first write F(s)= $\frac{1}{(s-a)(s-b)}=\frac{1}{(a-b)}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)$. Therefore, by the linearity, we obtain $\frac{1}{a-b}(e^{at}-e^{bt}) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{a-b}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)$. Thus, f(t)= $\frac{1}{a-b}(e^{at}-e^{bt})$.

2.2. **Laplace transform of the derivative of f(t):**

**Theorem 1:** Let f be differentiable and exponentially bounded.

$$\text{If } f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s), \text{ then } f'(t) \overset{\mathscr{L}}{\longleftrightarrow} sF(s)-f(0) \text{ for } s>0.$$

PROOF: $(\mathscr{L}f')(s) = \int_0^\infty f'(t)e^{-st}dt = e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt$ =sF(s)-f(0). (qed)

If the second derivative $f''$ exists, applying the above formula for $f'$ we obtain that

$$f''(t) \overset{\mathscr{L}}{\longleftrightarrow} s^2F(s)-sf(0)-f'(0).$$

Generally, by induction we have the following theorem.

**Theorem 2:** In case the $n^{th}$ derivative of f exists and exponentially bounded, we obtain the formula

$$f^{(n)}(t) \overset{\mathscr{L}}{\longleftrightarrow} s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0).$$

**Examples:** 1) For $f(t)=t^2$ we find F= $\mathscr{L}$f. To do so, we observe that $f'(t)=2t$; $f''(t)=2$.

We already have $2=f''(t) \overset{\mathscr{L}}{\longleftrightarrow} 2/s$. Therefore, $2/s = s^2F(s)-sf(0)-f'(0)$. It follows that $F(s)=2/s^3$.

2) Similarly, we easily obtain that $\sin wt \overset{\mathscr{L}}{\longleftrightarrow} \frac{w}{s^2+w^2}$.

**2.3. Laplace transform of the integral of f(t):**

**Theorem 3:** If $f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s)$, then $\int_0^t f(u)du \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s}F(s)$.

PROOF: Put g(t)= $\int_0^t f(u)du$. Then, $g'(t)=f(t)$ and g(0)=0. Let F = $\mathscr{L}$f and G= $\mathscr{L}$g. By Theorem 1, we obtain that $g'(t) \overset{\mathscr{L}}{\longleftrightarrow} sG(s)-g(0)=F(s)$. Therefore, G(s)=F(s)/s. (qed)

**Example:** For F(s) = $\frac{1}{s(s^2+w^2)}$ let find f(t). We already have $\frac{\sin wt}{w} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s^2+w^2}$.

By Theorem 3, we then derive $\int_0^t \frac{\sin wu}{w}du \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s(s^2+w^2)}$. Thus,

$$\frac{1-\cos wt}{w^2} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s(s^2+w^2)}.$$

**2.4. Inverse Laplace transform of the derivative of F(s):**

For $f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s)$ we can easily prove that $-tf(t) \overset{\mathscr{L}}{\longleftrightarrow} F'(s)$.

**Example:** Since $\sin wt \overset{\mathscr{L}}{\longleftrightarrow} \frac{w}{s^2+w^2}$ we have, by the above formula, that

$$t\sin wt \overset{\mathscr{L}}{\longleftrightarrow} \frac{2sw}{(s^2+w^2)^2}.$$

**2.5. Inverse Laplace transform of the integral of F(s):**

For $f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s)$ it can be proved that $\frac{f(t)}{t} \overset{\mathscr{L}}{\longleftrightarrow} \int_s^\infty F(u)du$.

**Example:** Let compute the inverse Laplace transform of G($s$) = $\ln(1+\frac{w^2}{s^2})$. To do that, we first write $\ln(1+\frac{w^2}{s^2})=-\int_s^\infty d\ln(1+\frac{w^2}{u^2})=\int_s^\infty \frac{2w^2}{u(u^2+w^2)}du$. Since $\frac{1-\cos wt}{w^2} \xleftrightarrow{\mathscr{L}} \frac{1}{s(s^2+w^2)}$ and applying the above formula we obtain $\frac{2(1-\cos wt)}{t} \xleftrightarrow{\mathscr{L}} \ln(1+\frac{w^2}{s^2})$.

**2.6. Shifting properties:**

(1) *s-shifting*: For $f(t) \xleftrightarrow{\mathscr{L}} F(s)$ we have that $e^{s_0 t}f(t) \xleftrightarrow{\mathscr{L}} F(s-s_0)$

**Example:** From $\sin wt \xleftrightarrow{\mathscr{L}} \frac{w}{s^2+w^2}$ we obtain that $e^{s_0 t}\sin wt \xleftrightarrow{\mathscr{L}} \frac{w}{(s-s_0)^2+w^2}$

(2) *t-shifting:* If we shift the function f(t), t≥0, to the right (i.e., we replace t by t-a for some a>0), then we encounter a problem that the function f(t-a) is no longer defined for a>t≥0. To come over this problem, we put $\tilde{f}(t)=\begin{cases}0 & if\ 0\le t<a\\ f(t-a) & if\ t\ge a\end{cases}$.

using the step function $u(t)=\begin{cases}0\ if\ t<0\\ 1\ if\ t\ge 0\end{cases}$ we can rewrite $\tilde{f}(t)=f(t-a)u(t-a)$. Then, for $f(t) \xleftrightarrow{\mathscr{L}} F(s)$ it can be proved that $f(t-a)u(t-a) \xleftrightarrow{\mathscr{L}} e^{-sa}F(s)$.

**Example:** Let compute the inverse Laplace transform of $\frac{e^{-2s}}{s^3}$. From the relation $t^2/2 \xleftrightarrow{\mathscr{L}} 1/s^3$ we can derive that $\frac{(t-2)^2}{2}u(t-2) \xleftrightarrow{\mathscr{L}} \frac{e^{-2s}}{s^3}$.

## 3. Convolution

**3.1. Lemma:** Let $f(t)$ and $g(t)$, $t\ge 0$, be piecewise continuous and exponentially bounded functions. Then, the function $h(t)=\int_0^t f(u)g(t-u)du$ is also exponentially bounded.

PROOF: Since $f(t)$ and $g(t)$ are piecewise continuous and exponentially bounded, we can estimate $|h(t)|\le \int_0^t M_1 e^{\gamma_1 u}e^{\gamma_2(t-u)}du=\frac{M_1M_2}{|\gamma_1-\gamma_2|}|e^{t\gamma_1}-e^{t\gamma_2}|=\frac{2M_1M_2}{|\gamma_1-\gamma_2|}e^{t\max\{\gamma_1,\gamma_2\}}$. Therefore, $h(t)$ is exponentially bounded.

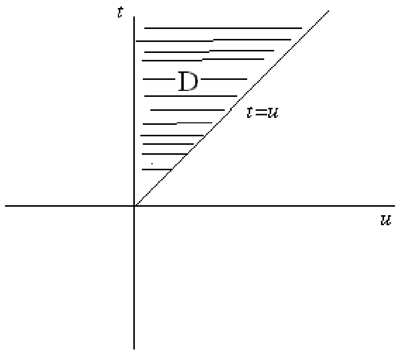
**3.2. Definition:** Let $f(t)$ and $g(t)$, $t\ge 0$, be piecewise continuous and exponentially bounded functions. Then the function $h(t)=\int_0^t f(u)g(t-u)du$ is called convolution of $f$ and $g$. Also, we denote by $h=f*g$. So, $h(t)=(f*g)(t)$. However, sometimes, physically we write $h(t)=f(t)*g(t)$.

**3.3. Theorem:** Let $f(t)$ and $g(t)$, $t\ge 0$, be piecewise continuous and exponentially bounded functions. Suppose $F=\mathscr{L}f$ and $G=\mathscr{L}g$. Then, $f(t)*g(t) \xleftrightarrow{\mathscr{L}} F(s)G(s)$.
Shortly, one can say that Laplace transform turns a convolution to a normal product.

PROOF: Let compute $\int_0^\infty (f*g)(t)e^{-st}dt=\int_0^\infty\int_0^t f(u)g(t-u)e^{-st}dudt=\int_0^\infty\int_u^\infty f(u)g(t-u)e^{-st}dtdu$

$$=\left(\int_0^\infty f(u)e^{-su}du\right)G(s) = F(s)G(s)$$

here, we used Fubini's Theorem for the domain described by following figure:




**Example:** Let compute $\mathscr{L}^{-1}\left(\frac{1}{(s^2+w^2)^2}\right)$. Since we already have $\frac{\sin wt}{w} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s^2+w^2}$,

using the convolution property we obtain $\frac{\sin wt}{w} * \frac{\sin wt}{w} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{(s^2+w^2)^2}$.

**3.4. Some other properties:**

1) Associative: $(f*g)*k = f*(g*k)$

*2)* Commutative: $f*g = g*f$

3) Distributive: $f*(g+k) = f*g + f*k$

## 4. Applications to Differential Equations

We have the following algorithm of using Laplace transform to solve differential equations of the order n: $f(t, y, \ldots, y^{(n)}) = r(t)$.

***Step 1:*** Apply the Laplace transform to both sides of the differential equation to obtain the simpler equation called subsidiary equation.
***Step 2:*** Solve the subsidiary equation.
***Step 3:*** Apply the inverse Laplace transform to obtain the solution of the original differential equation.

**Example 1:** $y''-y=t$; with the initial conditions $y(0)=1$; $y'(0)=1$;
Applying Laplace transform to the above equation and putting $Y = \mathscr{L}y$ we obtain

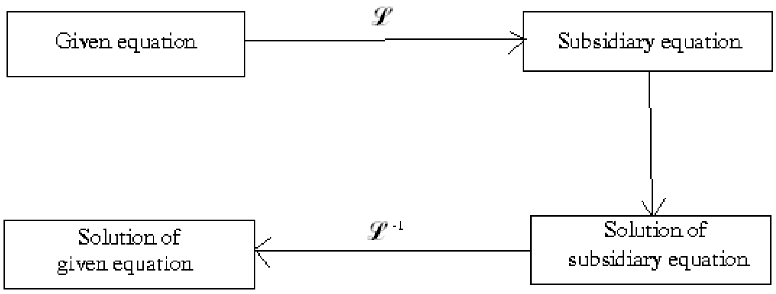
$$s^2Y(s)-sy(0)-y'(0)-Y(s)=1/s^2$$

$$\Leftrightarrow Y(s)=\frac{1}{s-1}+\frac{1}{s^2(s^2-1)}=\frac{1}{s-1}+\frac{1}{s^2-1}-\frac{1}{s^2}$$

Using the table of Laplace transforms we easily obtain that

$$y(t)=\mathscr{L}^{-1}Y=e^t+\sinh t-t.$$

Before continuing with further examples of applications of Laplace transform, we now introduce here a scheme for solving a differential equation using Laplace transform:




**Example 2:** Consider the general linear second-order differential equation with constant coefficients:

$$y'' + ay' + by=r(t); \text{ with initial conditions } y(0) = y_0, y'(0) = y_1. \quad (4.1)$$

Applying Laplace transform to both sides of the given equation and putting $Y= \mathscr{L}y$, $R=\mathscr{L}r$, we obtain the subsidiary equation: $(s^2 + as + b)Y(s)=R(s) + (s+a)y_0 + y_1$. Therefore, we have that the solution of subsidiary equation is

$$Y(s) = \frac{R(s)+(s+a)y_0+y_1}{s^2+as+b}.$$

Therefore, we obtain the solution of the given differential equation by taking the inverse Laplace transform of Y(s), i.e., the solution is $y= \mathscr{L}^{-1}Y$.

We now put $Q(s) = \frac{1}{s^2+as+b}$ and call it the transfer function. This name comes from the fact that, for some (mechanic or electric) systems, the function r(t) in equation (4.1) is called the input and the solution y(t) is called the output of the system, and in the special case when y(0)=0 and y'(0)=0, then Y(s)=R(s)Q(s). Therefore, $Q(s)=\mathscr{L}(\text{output})/\mathscr{L}(\text{input})$ explaining the name of Q(s). Also, in this case, the output is $y(t)=r(t)*q(t)$, where q(t) is inverse Laplace transform of Q(s).

# Tables of Laplace Transform:

## Table 1: General Formulae

| Formula | Name, Comments |
|---|---|
| $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ <br> $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | Definition of Transform <br> Inverse Transform |
| $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ | Linearity |
| $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ <br> $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ <br> $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$ | Differentiation of Function |
| $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$ | Integration of Function |
| $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ <br> $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$ | $s$-Shifting (1st Shifting Theorem) |
| $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ <br> $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ | $t$-Shifting (2nd Shifting Theorem) |
| $\mathcal{L}\{tf(t)\} = -F'(s)$ | Differentiation of Transform |
| $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$ | Integration of Transform |
| $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau = \int_0^t f(t-\tau)g(\tau)\, d\tau$ <br> $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ | Convolution |
| $\mathcal{L}(f) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$ | $f$ Periodic with Period $p$ |

**Table 2: Laplace Transform**

| | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ |
|---|---|---|
| 1 | $1/s$ | $1$ |
| 2 | $1/s^2$ | $t$ |
| 3 | $1/s^n, \quad (n = 1, 2, \cdots)$ | $t^{n-1}/(n-1)!$ |
| 4 | $1/\sqrt{s}$ | $1/\sqrt{\pi t}$ |
| 5 | $1/s^{3/2}$ | $2\sqrt{t/\pi}$ |
| 6 | $1/s^a \quad (a > 0)$ | $t^{a-1}/\Gamma(a)$ |
| 7 | $\dfrac{1}{s-a}$ | $e^{at}$ |
| 8 | $\dfrac{1}{(s-a)^2}$ | $te^{at}$ |
| 9 | $\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$ | $\dfrac{1}{(n-1)!} t^{n-1}e^{at}$ |
| 10 | $\dfrac{1}{(s-a)^k} \quad (k > 0)$ | $\dfrac{1}{\Gamma(k)} t^{k-1}e^{at}$ |
| 11 | $\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$ | $\dfrac{1}{(a-b)}(e^{at} - e^{bt})$ |
| 12 | $\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$ | $\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$ |
| 13 | $\dfrac{1}{s^2 + \omega^2}$ | $\dfrac{1}{\omega} \sin \omega t$ |
| 14 | $\dfrac{s}{s^2 + \omega^2}$ | $\cos \omega t$ |
| 15 | $\dfrac{1}{s^2 - a^2}$ | $\dfrac{1}{a} \sinh at$ |
| 16 | $\dfrac{s}{s^2 - a^2}$ | $\cosh at$ |
| 17 | $\dfrac{1}{(s-a)^2 + \omega^2}$ | $\dfrac{1}{\omega} e^{at} \sin \omega t$ |
| 18 | $\dfrac{s-a}{(s-a)^2 + \omega^2}$ | $e^{at} \cos \omega t$ |
| 19 | $\dfrac{1}{s(s^2 + \omega^2)}$ | $\dfrac{1}{\omega^2}(1 - \cos \omega t)$ |
| 20 | $\dfrac{1}{s^2(s^2 + \omega^2)}$ | $\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$ |
| 21 | $\dfrac{1}{(s^2 + \omega^2)^2}$ | $\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$ |

**Table 3: Laplace Transform (continued)**

| | $F(s) = \mathcal{L}\{f(t)\}$ | $f(t)$ |
|---|---|---|
| 22 | $\dfrac{s}{(s^2+\omega^2)^2}$ | $\dfrac{t}{2\omega}\sin \omega t$ |
| 23 | $\dfrac{s^2}{(s^2+\omega^2)^2}$ | $\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$ |
| 24 | $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ $(a^2 \neq b^2)$ | $\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$ |
| 25 | $\dfrac{1}{s^4+4k^4}$ | $\dfrac{1}{4k^3}(\sin kt \cosh kt - \cos kt \sinh kt)$ |
| 26 | $\dfrac{s}{s^4+4k^4}$ | $\dfrac{1}{2k^2}\sin kt \sinh kt$ |
| 27 | $\dfrac{1}{s^4-k^4}$ | $\dfrac{1}{2k^3}(\sinh kt - \sin kt)$ |
| 28 | $\dfrac{s}{s^4-k^4}$ | $\dfrac{1}{2k^2}(\cosh kt - \cos kt)$ |
| 29 | $\sqrt{s-a} - \sqrt{s-b}$ | $\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$ |
| 30 | $\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$ | $e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2}t\right)$ |
| 31 | $\dfrac{1}{\sqrt{s^2+a^2}}$ | $J_0(at)$ |
| 32 | $\dfrac{s}{(s-a)^{3/2}}$ | $\dfrac{1}{\sqrt{\pi t}} e^{at}(1+2at)$ |
| 33 | $\dfrac{1}{(s^2-a^2)^k}$ $(k > 0)$ | $\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$ |
| 34 | $e^{-as}/s$ | $u(t-a)$ |
| 35 | $e^{-as}$ | $\delta(t-a)$ |

| | | |
|---|---|---|
| 36 | $\frac{1}{s} e^{-k/s}$ | $J_0(2\sqrt{kt})$ |
| 37 | $\frac{1}{\sqrt{s}} e^{-k/s}$ | $\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$ |
| 38 | $\frac{1}{s^{3/2}} e^{k/s}$ | $\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$ |
| 39 | $e^{-k\sqrt{s}}$ $(k > 0)$ | $\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$ |
| 40 | $\frac{1}{s} \ln s$ | $-\ln t - \gamma$ $(\gamma \approx 0.5772)$ |
| 41 | $\ln \frac{s-a}{s-b}$ | $\frac{1}{t}(e^{bt} - e^{at})$ |
| 42 | $\ln \frac{s^2+\omega^2}{s^2}$ | $\frac{2}{t}(1 - \cos \omega t)$ |
| 43 | $\ln \frac{s^2-a^2}{s^2}$ | $\frac{2}{t}(1 - \cosh at)$ |
| 44 | $\arctan \frac{\omega}{s}$ | $\frac{1}{t} \sin \omega t$ |

## Problems

In each of Problems 1 through 10 find the inverse Laplace transform of the given function.

1. $\frac{3}{s^2+4}$
2. $\frac{4}{(s-1)^3}$
3. $\frac{2}{s^2+3s-4}$
4. $\frac{3s}{s^2-s-6}$
5. $\frac{2s+2}{s^2+2s+5}$
6. $\frac{2s-3}{s^2-4}$
7. $\frac{2s+1}{s^2-2s+2}$
8. $\frac{8s^2-4s+12}{s(s^2+4)}$
9. $\frac{1-2s}{s^2+4s+5}$
10. $\frac{2s-3}{s^2+2s+10}$

In each of Problems 11 through 23 use the Laplace transform to solve the given initial value problem.

11. $y'' - y' - 6y = 0; \quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$
13. $y'' - 2y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' - 2y = 0; \quad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = -1$
17. $y^{iv} - 4y''' + 6y'' - 4y' + y = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
18. $y^{iv} - y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{iv} - 4y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \quad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t; \quad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t}; \quad y(0) = 2, \quad y'(0) = -1$

In each of Problems 24 through 36 find the solution of the given initial value problem.

24. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & \pi/2 \le t < \infty \end{cases}$

25. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \quad y'(0) = 1; \quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \text{ and } t \ge 2\pi \end{cases}$

26. $y'' + 4y = \sin t - u(t-2\pi)\sin(t - 2\pi)$; $y(0) = 0$, $y'(0) = 0$
27. $y'' + 4y = \sin t + u(t-\pi)\sin(t - \pi)$; $y(0) = 0$, $y'(0) = 0$
28. $y'' + 3y' + 2y = f(t)$; $y(0) = 0$, $y'(0) = 0$; $f(t) = 1$ for $0 \le t < 10$ and $f(t) = 0$ for $t \ge 10$
29. $y'' + 3y' + 2y = u(t-2)$; $y(0) = 0$, $y'(0) = 1$
30. $y'' + y = u(t-3\pi)$; $y(0) = 1$, $y'(0) = 0$
31. $y'' + y' + \frac{5}{4}y = t - u(t - \pi/2)(t - \pi/2)$; $y(0) = 0$, $y'(0) = 0$
32. $y'' + y = g(t)$; $y(0) = 0$, $y'(0) = 1$; $g(t) = t/2$ for $0 \le t < 6$ and $g(t) = 3$ for $t \ge 6$,
33. $y'' + y' + \frac{5}{4}y = g(t)$; $y(0) = 0$, $y'(0) = 0$; $g(t) = \sin t$ for $0 \le t < \pi$ and $g(t) = 0$ for $t \ge \pi$
34. $y'' + 4y = u(t-\pi) - u(t-3\pi)$; $y(0) = 0$, $y'(0) = 0$
35. $y^{(4)} - y = u(t-1) - u(t-2)$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$
36. $y^{(4)} + 5y'' + 4y = 1 - u(t-\pi)$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

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