Recovery of the fractional diffusion equation from a single boundary measurement

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Abstract

Consider a fractional diffusion process on a finite length rod

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u(x,t) = u_{xx}(x,t) - q(x)u(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = a(t), \\ u(x,0) = 0. \end{cases} \tag{1}$$

Here ${}_0^C\mathcal{D}_t^\alpha u(x,t)$, $0 < \alpha < 1$, denotes the Caputo derivative

$$ {}_0^C\mathcal{D}_t^\alpha u(x,t) = \int_0^t \frac{(t-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \frac{\partial}{\partial \xi} u(x,\xi)\, d\xi. $$

We are concerned with the recovery of the diffusion coefficient $q(x) \in L_1(0,1)$ from the measurement of $u_x(1,t) = b(t),\ t \in (0,\infty)$, at one end of the rod only. This problem is close in spirit with boundary control, where a is a given input on the boundary and we can observe its response b also on the boundary. However the boundary control method is usually applied to wave equations because of their finite wave speed propagation.

We prove that we can uniquely recover $q(x)$ from a single boundary measurement $b(t)$ and provide a constructive procedure for its recovery. The algorithm is based on the well known Gelfand-Levitan-Gasymov inverse spectral theory of Sturm-Liouville operators. More precisely, we have

Theorem 1. *Let boundary condition $u(1,t) = a(t)$ be a nontrivial nonnegative bounded function with $a(0) = 0$. Then the single measurement of $u_x(1,t) = b(t)$, $t \in (0,\infty)$ determines $q(x) \in L_1(0,1)$ uniquely.*

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