Inverse Heat Equation

Vu Kim Tuan
University of West Georgia, Carrollton, Georgia, USA
email: vu@westga.edu

Abstract We are concerned with an inverse problem associated with the linear parabolic equation defined by

$$(1)\quad \begin{cases} \dfrac{\partial}{\partial t}u(x,t) = \sum\limits_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j}u(x,t)\right) - q(x)u(x,t), & x \in \Omega \subset \mathbb{R}^n,\ n \geq 2,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = \psi(x) \in L^2(B), & \overline{\Omega} \subset B \end{cases}$$

which describes the evolution of heat in an anisotropic media with thermal coefficients $a_{ij}(x)$ and an internal heat sink coefficient $q(x)$. Assume that a body of an unknown composition and shape Ω is enclosed in a black box B and cannot be seen. The box B is given an initial temperature $\psi(x)$ and is left to cool. Assume that we can measure its temperature at a single point $x = b$, i.e., $u(b,t)$, from inside the box B, or heat flux $\frac{\partial u(b,t)}{\partial n}$, at a single boundary point $x = b \in \partial\Omega$, for different initial temperature profiles $u(x,0) = \psi(x)$. We show that we can determine uniquely the composition of the body Ω, its geometry, the diffusion thermal coefficients $a_{ij}(x)$, and the internal heat sink coefficient $q(x)$.

First consider the case when $a_{ij}(x) = 1$ and the domain $\Omega \subset \mathbb{R}^n$ is known. The idea is to reconstruct the first eigenfunction φ_1 by recovering its Fourier coefficients from the measurements of heat flux at a single boundary point $b \in \partial\Omega$ under different initial temperatures. Once φ_1 is reconstructed, the only unknown $q(x)$ is recovered from the facts that $-\Delta\varphi_1(x) + q(x)\varphi_1(x) = \lambda_1\varphi_1(x)$ and $\varphi_1(x) \neq 0$, for all $x \in \Omega$. To extend the idea to equation (1) to reconstruct all the coefficients a_{ij} and q we need more than just the first eigenfunction, but at least $n(n+1)/2 + 1$ (the number of unknown coefficients) eigenfunctions. The key idea is to prove that we can extract infinitely many eigenfunctions from measurement at any given point $b \in \Omega$. After reconstructing enough eigenfunctions, the next step is to show that we can solve for the unknowns $a_{ij}(x)$ and $q(x)$ uniquely from systems of linear equations. This question reduces to showing that certain generalized Wronskians of eigenfunctions are nonsingular.

In one-dimensional case we show that only finitely many measurements at end points are enough to determine $q(x)$ uniquely.

This is a joint work with A. Boumenir and Nguyen Hoang.