Long-time behavior of solutions to some equations in fluid mechanics

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Some basic problems for PDEs

$$\begin{cases} \partial_t u = \mathcal{F}(t, u(t)), t > \tau, \\ u(\tau) = u_\tau. \end{cases}$$

- Well-posedness
- Regularity of solutions
- Long-time behavior of solutions: Stability theory + attractors theory
- Control theory:
  - Controllability
  - Optimal control
  - Stabilization

Process

$$\begin{cases} \partial_t u = \mathcal{F}(t, u(t)), t > \tau, \\ u(\tau) = u_\tau \in X. \end{cases} \tag{1.1}$$

Assume that for each $u_\tau \in X$ given, problem (1.1) has a unique global solution $u(t; \tau, u_\tau)$. Putting $U(t,\tau)u_\tau := u(t;\tau,u_\tau)$, we get a process $U(t,\tau)$ on X, which is called the process associated to problem (1.1).

A process on X is a family of two-parameter mappings $\{U(t,\tau)\}$ in X having the following properties:

$$U(\tau,\tau) = Id \text{ for all } \tau \in \mathbb{R},$$

$$U(t,r)U(r,\tau) = U(t,\tau) \text{ for all } t \geq r \geq \tau.$$

Theory of attractors

Study the long-time behavior of solutions: theory of attractors

- Autonomous PDEs: Solution semigroup $S(t) : u_0 \mapsto u(t)$.
  Since 1980s: Theory of global attractors.
- Non-autonomous PDEs: The associated process:

$$U(t,\tau) : u_\tau \mapsto U(t,\tau)u_\tau = u(t).$$

  - Theory of uniform attractors: Chepyzhov-Vishik (1994);
  - Theory of pullback attractors: Caraballo-Łukaszewicz-Real (2006).
- Advantages of pullback attractors:
  - allow to handle a larger class of time-dependent external forces;
  - (Usually) have a finite fractal dimension;
  - are also valid for random dynamical systems.

Definition of pullback $\mathcal{D}$-attractors

Suppose that $\mathcal{B}(X)$ is the family of all nonempty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.
A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback $\mathcal{D}$-attractor for the process $U(t,\tau)$ if

1. $A(t)$ is compact for all $t \in \mathbb{R}$;
2. $\hat{\mathcal{A}}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(t), \text{ for all } t \geq \tau$;
3. $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,
$$\lim_{\tau \to -\infty} dist(U(t,\tau)D(\tau), A(t)) = 0, \text{ for all } \hat{\mathcal{D}} \in \mathcal{D}, \text{ and all } t \in \mathbb{R};$$
4. If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Some basic concepts

- The process $U(t,\tau)$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$, and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.
- The family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process $U(t,\tau)$ if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \leq t$ such that

$$\bigcup_{\tau \leq \tau_0} U(t,\tau)D(\tau) \subset B(t).$$

Existence of pullback $\mathcal{D}$-attractors

Let $\{U(t,\tau)\}$ be a continuous process on X such that

1. $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact;
2. there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$.

Then $\{U(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$, and

$$A(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau)B(\tau)}.$$

Some references

1. T. Caraballo, G. Łukaszewicz and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.* 64 (2006), 484-498.
2. A.N. Carvalho, J.A. Langa and J.C. Robinson, *Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems*, Applied Mathematical Sciences 182, Springer, Berlin, 2013.

Navier-Stokes-Voigt equations

Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\partial\Omega$. Consider the following problem

$$\begin{cases} \partial_t u - \nu\Delta u - \alpha^2 \Delta u_t + (u \cdot \nabla)u + \nabla p & = f, \ x \in \Omega, t > \tau, \\ \nabla \cdot u & = 0, \ x \in \Omega, t > \tau, \\ u(x,t) & = 0, \ x \in \partial\Omega, t > \tau, \\ u(x,\tau) & = u_0(x), \ x \in \Omega, \end{cases} \tag{2.1}$$

where $u = u(x,t) = (u_1, \ldots, u_d)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, and u_0 is the initial velocity.

- Meaning: a model of linear viscoelastic fluids (Oskolkov, 1973) + a regularization of the Navier-Stokes equations (Titi et. al., 2006).
- Difficulties: the system is only weakly dissipative + the domain may be unbounded.

Hypotheses

(H1) The domain Ω can be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^d$ without any regularity assumption on its boundary $\partial\Omega$, provided that the Poincaré inequality holds on Ω: There exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla\phi|^2 dx, \ \ \forall\phi \in H_0^1(\Omega).$$

(H2) $f \in L^2_{loc}(\mathbb{R}; V')$ such that

$$\int_{-\infty}^0 e^{\sigma s} \|f(s)\|_{V'}^2 ds < +\infty,$$

where $\sigma = \frac{\lambda_1 \nu}{1+\alpha^2\lambda_1}$.

Function spaces

Let $(L^2(\Omega))^d$ and $(H_0^1(\Omega))^d$ be endowed with the inner products

$$(u,v) = \int_\Omega u.v dx, \quad u,v \in (L^2(\Omega))^d,$$

$$((u,v)) = \int_\Omega \sum_{j=1}^d \nabla u_j \cdot \nabla v_j dx, \; u,v \in (H_0^1(\Omega))^d,$$

and norms $|u|^2 = (u,u), \|u\|^2 = ((u,u))$.
Let

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^d \; : \; \nabla \cdot u = 0 \right\}.$$

Denote by H the closure of $\mathcal{V}$ in $(L^2(\Omega))^d$, and by V the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^d$. It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V', and $\langle .,. \rangle$ for the duality pairing between V and V'.

Operators

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{d} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V$, then

$$b(u,v,w) = -b(u,w,v).$$

Hence

$$b(u,v,v) = 0, \ \forall u, v \in V.$$

Set $A : V \to V'$ by $\langle Au, v\rangle = ((u,v))$, $B : V \times V \to V'$ by $\langle B(u,v), w\rangle = b(u,v,w)$, $Bu = B(u,u)$. Then $Au = -P\Delta u$, for all $u \in D(A)$, where P is the ortho-projector from $(L^2(\Omega))^d$ onto H.

Basic inequalities

Using Hölder's inequality and Ladyzhenskaya's inequality:

$$|u|_{L^4} \le c_0 |u|^{d/4} |\nabla u|^{1-d/4}, \ \ \forall u \in H_0^1(\Omega),$$

and the interpolation inequalities, one can prove the following

- If $d = 2, 3$, then

  $$|b(u, v, w)| \le C|u|^{1-d/2} \|u\|^{d/4} |v|^{1-d/4} \|v\|^{d/4} \|w\|, \ \forall u, v, w \in V.$$

  In particular,

  $$|b(u, u, v)| \le C|u|^{2-2/d} \|u\|^{d/2} \|v\|, \ \forall u, v \in V.$$

- Let $u \in L^2(\tau, T; V)$. Then the function Bu defined by

  $$(Bu(t), v) = b(u(t), u(t), v), \forall u \in V, \ a.e. \ t \in [\tau, T],$$

  belongs to $L^1(\tau, T; V')$.

Definition of weak solutions

Given $u_0 \in V$. A function u is called a **weak solution** to problem (2.1) on the interval (τ, T) if

$$\begin{cases} u \in C([\tau, T]; V), \; du/dt \in L^2(\tau, T; V), \\ \dfrac{d}{dt} u(t) + \nu A u(t) + \alpha^2 A u'(t) + B(u(t), u(t)) = f(t) \text{ in } V', \text{for a.e.} t, \\ u(\tau) = u_0. \end{cases}$$

Existence theorem

Suppose that $u_0 \in V$ is given and assumptions $(H1) - (H2)$ hold. Then, for any $\tau \in \mathbb{R}$, $T > \tau$ given, problem (2.1) has a unique weak solution u on (τ, T). Moreover, the map $u_0 \mapsto u(t)$ is continuous on V for all $t \in [\tau, T]$.

Sketch of the proof

Using the Galerkin approximation + the Compactness Lemma

- **Step 1:** Construct the approximate solutions u_m

$$\begin{cases} \dfrac{d}{dt}u_m(t) + \nu A u_m(t) + \alpha^2 A u_m'(t) + P_m B u_m(t) = P_m f(t) \text{ in } V', \\ u_m(\tau) = P_m u_0. \end{cases}$$

- **Step 2:** Establish some *a priori* estimates for u_m

$$|u^m(t)|^2 + \nu \int_\tau^t |\nabla u^m(s)|^2 ds + \alpha^2 |\nabla u^m(t)|^2 \\ \leq \frac{1}{\nu} \|f\|^2_{L^2(\tau,T;V')} + |u_0|^2 + \alpha^2 |\nabla u_0|^2.$$

- **Step 3:** Passage to the limit
- **Step 4:** Prove the uniqueness and continuous dependence of the weak solutions on the initial data.

The associated process

Thanks to the existence theorem, one can define a continuous process $U(t,\tau)$ in V by

$$U(t,\tau)u_0 = u(t;\tau,u_0),\ \tau \le t, u_0 \in V,$$

where $u(t) = u(t;\tau,u_0)$ is the unique weak solution of problem (2.1) with the initial datum $u(\tau) = u_0$.

Lemma. Let $\{u_{0_n}\}$ be a sequence in V converging weakly in V to an element $u_0 \in V$. Then

$$U(t,\tau)u_{0_n} \rightharpoonup U(t,\tau)u_0 \quad \text{weakly in } V, \quad \text{for all } \tau \le t,$$

$$U(t,\tau)u_{0_n} \rightharpoonup U(t,\tau)u_0 \quad \text{weakly in } L^2(\tau,T;V), \quad \text{for all } \tau < T.$$

Existence of a pullback $\mathcal{D}$-attractor

Let $\mathcal{R}_\sigma$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^{\sigma t} r^2(t) = 0,$$

where $\sigma = \frac{\lambda_1 \nu}{1+\alpha^2\lambda_1}$, and $\mathcal{D}_\sigma$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(H)$ such that $D(t) \subset B(0, \hat{r}(t))$, for some $\hat{r}(t) \in \mathcal{R}_\sigma$, where $B(0, r)$ denotes the close ball in V, centered at zero with radius r.

Theorem. Suppose that conditions $(H1) - (H2)$ hold. Then, there exists a unique pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}_\alpha = \{A_\alpha(t) : t \in \mathbb{R}\}$ for the process $\{U(t, \tau)\}$ associated to problem (2.1).

Sketch of the proof

We will check the two conditions in the abstract theorem:

1. $U(t,\tau)$ has a family $\hat{B}$ of pullback $\mathcal{D}_\sigma$-absorbing sets

$$\|u(t)\|^2 \leq \frac{1}{\alpha^2} e^{-\sigma(t-\tau)}[u_0]_2^2 + \frac{e^{-\sigma t}}{\nu\alpha^2}\int_\tau^t e^{\sigma r}\|f(r)\|_*^2 dr,$$

   where $[u]_2^2 := |u|^2 + \alpha^2\|u\|^2$. Denote $R_\sigma^2(t) := \dfrac{2e^{-\sigma t}}{\nu}\int_{-\infty}^t e^{\sigma s}\|f(s)\|_*^2 ds$, then $\{B_\sigma(t) : t \in \mathbb{R}\}$ is a family of bounded pullback $\mathcal{D}_\sigma$-absorbing sets.

2. $U(t,\tau)$ is pullback $\mathcal{D}_\sigma$-asymptotically compact: Using the energy equation method introduced by J.M. Ball, *Disc. Cont. Dyn. Syst.* (2004).

Estimates of the fractal dimension

We suppose furthermore that

$$f \in L^{\infty}(-\infty, T^*; V'), \quad \text{for some } T^* \in \mathbb{R}. \tag{2.2}$$

Theorem. Suppose that conditions $(H1), (H2)$ and (2.2) hold. Then, the pullback $\mathcal{D}_\sigma$-attractor $\hat{A}_\alpha = \{A_\alpha(t) : t \in \mathbb{R}\}$ of the process $U(t, \tau)$ associated to problem (2.1) satisfies

$$d_F(A_\alpha(t)) \leq 2 + \frac{C(\lambda_1 + \alpha^2)^2 \|f\|^4_{L^\infty(-\infty, T^*; V')}}{\nu^6 \alpha^6 \sigma^2}, \quad \text{for all } t \in \mathbb{R}.$$

For more details, see Anh-Trang, *Proc. R. Soc. Edinb. Sect. A Math.* (2013).

Regularity of the pullback attractor for Navier-Stokes-Voigt equations

We assume that

(F) $f \in L^2_{loc}(\mathbb{R}; H)$, $f' \in L^2_{loc}(\mathbb{R}; V')$ such that

$$\int_{-\infty}^{0} e^{\frac{\sigma s}{3}} \|f(s)\|^2_{-1/2} ds < +\infty;$$

$$\int_{-\infty}^{0} e^{\sigma s} |f(s)|^2 ds < +\infty,$$

where $\sigma = \frac{\lambda_1 \nu}{1+\alpha^2 \lambda_1}$.

Theorem. The pullback $\mathcal{D}_\alpha$-attractor $\hat{A}_\alpha = \{A_\alpha(t) : t \in \mathbb{R}\}$ is compact in $(H^2(\Omega))^2 \cap V$ in the sense that for any fixed $t \in \mathbb{R}$, $A_\alpha(t)$ is a compact set in $(H^2(\Omega))^2 \cap V$.

Upper semicontinuity of pullback attractors for 2D Navier-Stokes-Voigt equations

- $\alpha = 0$: Navier-Stokes equations
- Pullback attractors in V for strong solutions to 2D Navier-Stokes equations:
  - bounded domain: Real et. al., *J. Diff. Equa.* (2012).
  - unbounded domain: Anh-Trang, *preprint* (2013).
- Upper semicontinuity of pullback attractors:
  Theorem. The family of pullback $\mathcal{D}_\alpha$- attractors $\hat{A}_\alpha = \{A_\alpha(t) : t \in \mathbb{R}\}$ for 2D Navier-Stokes-Voigt equations is upper semicontinuous in V at $\alpha = 0$, that is, for any $t \in \mathbb{R}$,

$$\lim_{\alpha \to 0} dist_V(A_\alpha(t), A(t)) = 0.$$

Existence, uniqueness and stability of stationary solutions

- Let $f \in V'$ be independent of time. A **stationary solution** to problem (2.1) is an element $u^* \in V$ such that

$$\nu((u^*, v)) + b(u^*, u^*, v) = \langle f, v \rangle, \ \forall v \in V.$$

- Assume that $f \in V'$ and

$$\nu^2 > \frac{c_0^2}{\lambda_1^{1/2}} \|f\|_*,$$

where c_0 is the best constant in Ladyzhenskaya's inequality. Then there exists a unique stationary solution to problem (2.1). Moreover, this stationary solution is exponentially stable:

$$|u(t) - u^*|^2 \leq e^{-\lambda t} |u_0 - u^*|^2.$$

Some generalizations of Navier-Stokes equations

- 3D convective Brinkman-Forchheimer equations (or tamed/damping Navier-Stokes equations)

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u + f(x,u) + \nabla p &= g, \\ \nabla \cdot u &= 0, \end{cases}$$

where $f(\cdot, u) \sim |u|^{p-1} u, p > 3$.

  - Global attractor (bounded domain): Kalantarov-Zelik, *Comm. Pure Appl. Anal.* (2012).
  - Pullback attractor (unbounded domain): Anh-Trang, *submitted*.

Some generalizations of Navier-Stokes equations (cont.)

- Kelvin-Voigt-Brinkman-Forchheimer equations

$$\begin{cases} u_t - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + f(x,u) + \nabla p & = g, \\ \nabla\cdot u & = 0, \end{cases}$$

  where $f(\cdot,u) \sim |u|^{p-1}u, p \geq 1$.

See Anh-Trang, *Nonlinear Anal.* (2013).

- 2D second grade fluid equations

$$\begin{cases} \partial_t(u - \alpha\Delta u) - \nu\Delta u + \text{curl}(u - \alpha\Delta u) \times u + \nabla p = f, x \in \Omega, t > \tau \\ \nabla \cdot u = 0,\ x \in \Omega, t > \tau, \\ u(x,t) = 0,\ x \in \partial\Omega, t > \tau, \\ u(x,\tau) = u_0(x),\ x \in \Omega. \end{cases}$$

Here the parameters α and ν are given positive constants and the initial datum u_0 satisfies the compatibility condition

$$\text{div} u_0 = 0 \ \text{ in } \ \Omega \quad \text{ and } \quad u_0 = 0 \ \text{ on } \ \partial\Omega.$$

- Fluids with pressure dependent viscossities

Some coupled systems in fluid mechanics

- the 2D Bénard problem (or Boussinesq system): a system with the Navier-Stokes equations for the velocity field coupled with a convection-diffusion equation for the temperature.

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f_u(x,t) + \alpha \vec{e_2}(T - T_r), \\ \partial_t T + (u \cdot \nabla) T - \kappa \Delta T = f_T(x,t), \\ \nabla \cdot u = 0, \end{cases}$$

See Anh-Son, *Math. Methods Appl. Sci.* (2013).

Some coupled systems in fluid mechanics (cont.)

- 2D magnetohydrodynamic (MHD) equations: a system with the Navier-Stokes equations for the velocity field coupled with a convection-diffusion equation for the magnetic fields

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \dfrac{1}{R_e}\Delta u + \nabla p + S\nabla(\dfrac{B^2}{2}) - S(B\cdot\nabla)B = f, \\ \dfrac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u + \dfrac{1}{R_m}\tilde{\text{curl}}(\text{curl B}) = 0, \\ \text{div}u = 0, \\ \text{div}B = 0. \end{cases}$$

- the two-phase flows: Cahn-Hilliard-Navier-Stokes system, Allen-Cahn-Navier-Stokes system, models for the nematic liquid crystal flows.

Further readings

1. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, 3nd edition, Amsterdam: North-Holland, 1984.
2. R. Temam, Some developments on Navier-Stokes equations in the second half of the 20th century, *Development of Mathematics 1950–2000*, 1049-1106, Birkhäuser, Basel, 2000.
3. J. Málek and K. Rajagopal, Mathematical issues concerning the Navier-Stokes equations and some of its generalizations, *Handbook of Differential Equations*. Vol. II, 371-459, Elsevier, 2005.
4. E. Feireisl and D. Pražák, *Asymptotic Behavior of Dynamical Systems in Fluid Mechanics*, Springer, 2010.
5. M. Petcu, R. Temam and M. Ziane, Some mathematical problems in geophysical fluid dynamics, *Handbook of Numerical Analysis*. Vol. XIV, 577-750, Elsevier, 2009.

Thank you for your attention!