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The Clarke Coderivative of the Frontier Map in a Multi-objective Optimal Control Problem

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Abstract
Motivated by our recent works on the efficient point multifunction in multi-objective parametric optimal control problems with nonconvex cost functions and control constrains, in this paper we study of the first-order behavior of the efficient point multifunction in a multi-objective parametric optimal control problem under nonlinear state equations. By establishing an abstract result on the Clarke coderivative of the frontier map of a multi-objective parametric mathematical programming problem, we derive a formula for computing the Clarke coderivative of the efficient point multifunction to a multi-objective parametric optimal control problem.

Keywords Multi-objective parametric optimal control problem · Efficient point multifunction · The Frontier map · Clarke normal cone · Clarke coderivative · Clarke tangent cone

Mathematics Subject Classification 34K35 · 49J53 · 90B50 · 90C31 · 93C15

1 Introduction

The class of multi-objective optimal control problems are important because they have many applications in economics, aerospace, multiobjective control design, environmental studies where we need to optimize many objectives (see [2, 3, 11–13, 27, 31, 38]). For a specific example, in transportation we want to reach to a destination as fast as possible while minimizing energy consumption, we need to use the model of two-objective optimal control (see, for instance [27]).

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Recently, by establishing an abstract result on the subdifferential of the frontier map in a multi-objective parametric optimization problem, Toan and Thuy [37] have obtained a formula for computing the Mordukhovich subdifferential of the frontier map to a multi-objective parametric optimal control problem with nonconvex objective functions, the *linear state equation* and the control constraint. Note that if the state equation in the optimal control problem is linear, then the graph of the constraint function in the optimization problem is convex. Then, we can compute the normal cone of the graph of the constraint function via normal cones of convex sets. So, we can use [37, Theorem 3.1] to obtain formulae for upper and lower-evaluation on the Mordukhovich subdifferential of the frontier map to a multi-objective parametric optimal control problem. However, the situation will be more complicated if the state equation is nonlinear because normal cone calculus of convex sets fail to apply.

The study of sensitivity analysis for multi-objective optimization problems as well as for multiobjective optimal control problems is a fundamental topic in variational analysis and optimization. There have been a lot of papers dealing with differentiability properties and subdifferentials of the frontier map (see [1, 7–10, 14, 18, 19, 30, 32, 33]). Normally, there are two approaches to study sensitivity analysis for optimization problems, either through the primal space or through the dual space. Via the concept of the contingent derivative in the primal space, several authors have studied the behavior of the frontier map in [1, 9, 18, 19, 30, 32, 33]. Using the notion of normal cones which is defined in dual space, authors [7, 8, 10, 14] have obtained sensitivity analysis results for mathematical programming problems with *functional constraints*.

In [37], we have obtained formulas for computing the Mordukhovich subdifferential of the frontier map in a multi-objective parametric mathematical programming problem with geometrical and functional constraints. Note that in [37], the functional constraint is defined via linear mappings. So, constraint sets of the multi-objective parametric mathematical programming problem are all convex. Hence, we can compute the normal cone of the constraint set through the intersection of two normal cones (see [37, Lemma 3.2]). But in this direction, we did not see formulas for computing the Clarke coderivative of the frontier map in a multi-objective parametric mathematical programming problem with *geometrical and functional constraints* where the functional constraint is defined via nonconvex mappings.

In this paper, we continue to study sensitivity analysis to multi-objective parametric optimal control problems with nonconvex objective functions, *nonlinear state equations* and control constraints by giving shaper formulas for computing the Clarke corderivative of the frontier map. In order to prove the main result, we first reduce the problem to a multi-objective parametric mathematical programming problem and establish formulae for upper and lower-evaluation on the Clarke corderivative of the frontier map via the normal cone of the constraint set, the Fréchet derivative of objective functions and constraint functions. Then, we apply the obtain results to derive formulas for computing the Clarke coderivative on the frontier map in a multi-objective parametric optimal control problem.

The paper is organized as follows. In Sect. 2, we sate the control problem and recall some auxiliary results. Formulae for upper and lower-evaluation on the Clarke corderivative of the frontier map to a specific mathematical programming problem is studied in Sect. 3. The last section establishes one theorem and one corollary on estimating/computing the Clarke corderivative of the frontier map to the multi-objective parametric optimal control problem. Section 4 also presents an example to illustrate the main result of this paper.

2 Problem Formulation and Auxiliary Results

For the convenience of the reader, we divide this section into three subsections. In the first subsection, we introduce the multi-objective parametric optimal control problem that we are interested in. The second subsection transforms the problem to a multi-objective parametric optimization problem under geometrical and functional constraints. In the last subsection, we recall some notions and facts from variational analysis and generalized differentiation, which are related to our problem.

2.1 Control Problem

A wide variety of problems in optimal control problem can be posed in the following form.

Determine a control vector $u \in L^p([0, 1], \mathbb{R}^m)$ and a trajectory $x \in W^{1,p}([0, 1], \mathbb{R}^n)$, $1 < p < \infty$, which solve

$$\mathrm{Min}_{\mathbb{R}^s_+} J(x, u, \theta), \tag{1}$$

with the state equation

$$\dot{x}(t) = \varphi\big(t, x(t)\big) + B(t)u(t) + T(t)\theta(t) \text{ a.e. } t \in [0, 1], \tag{2}$$

the initial value

$$x(0) = \alpha, \tag{3}$$

and the control constraint

$$u \in \mathcal{U}. \tag{4}$$

Here $W^{1,p}([0, 1], \mathbb{R}^n)$ is the Sobolev space consisting of absolutely continuous functions $x : [0, 1] \to \mathbb{R}^n$ such that $\dot{x} \in L^p([0, 1], \mathbb{R}^n)$. Its norm is given by

$$\|x\|_{1,p} = |x(0)| + \|\dot{x}\|_p.$$

The notations in (1)–(4) have the following meanings:

- x, u are the state variable and the control variable, respectively,
- $(\alpha, \theta) \in \mathbb{R}^n \times L^p([0, 1], \mathbb{R}^k)$ are parameters,
- $J(x, u, \theta) = \Big(J^1(x, u, \theta), J^2(x, u, \theta), \ldots, J^s(x, u, \theta)\Big)$
  $J^i(x, u, \theta) = g^i\big(x(1)\big) + \int_0^1 L^i\big(t, x(t), u(t), \theta(t)\big)dt,$

- $\varphi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $g^i : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $L^i : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \bar{\mathbb{R}}$ $(i = 1, 2, \ldots, s)$ are given functions,
- $B(t) = \big(b_{ij}(t)\big)_{n\times m}$ and $T(t) = \big(c_{ij}(t)\big)_{n\times k}$ are matrix-valued functions,
- $\mathcal{U}$ is a closed and convex set in $L^p([0,1], \mathbb{R}^m)$,
- $\mathrm{Min}_{\mathbb{R}^s_+} J(x, u, \theta)$ is the set of *efficient points* of

$$A := \{J(x, u, \theta) : (x, u, \theta) \text{ are satisfied (2)-(4) }\}$$

with respect to $\mathbb{R}^s_+$, that includes $y \in A$ such that $(y - \mathbb{R}^s_+) \cap A = \{y\}$. When $A = \emptyset$, we stipulate that $\mathrm{Min}_{\mathbb{R}^s_+} A = \emptyset$.

This type of problems are investigated in [6, 16, 17, 24–26, 34, 35, 40] and the references therein.

2.2 Reduction to a Parametric Optimization Problem

Put $X = W^{1,p}([0,1], \mathbb{R}^n)$, $U = L^p([0,1], \mathbb{R}^m)$, $\Theta = L^p([0,1], \mathbb{R}^k)$, $W = \mathbb{R}^n \times \Theta$. It is well known that X, U, Θ and W are Asplund spaces. For each $w = (\alpha, \theta) \in W$, we put

$$H(w) = \big\{(x, u) \in X \times U : \text{(2) and (3) are satisfied}\big\}, \tag{5}$$

and

$$K = X \times \mathcal{U}.$$

Then, the problem $(1) - (4)$ can be written in the following form:

$$\mathrm{Min}_{\mathbb{R}^s_+} \quad J(x, u, w), \quad \text{subject to } \; (x, u) \in H(w) \cap K. \tag{6}$$

Let $F : W \rightrightarrows \mathbb{R}^s$ be the multifunction given by

$$F(w) = (J \diamond H_K)(w) := \{J(x, u, w) : (x, u) \in H_K(w)\}, \tag{7}$$

where

$$H_K(w) = H(w) \cap K, \; \forall w \in W.$$

We put

$$\mathcal{F}(w) = \mathrm{Min}_{\mathbb{R}^s_+} F(w), \quad w \in W \tag{8}$$

and call $\mathcal{F} : W \rightrightarrows \mathbb{R}^s$ the *efficient point multifunction* or *the frontier map* of the problem $(1) - (4)$.

2.3 Some Facts from Variational Analysis and Generalized Differentiation

In this subsection, we recall some notions and facts from variational analysis and generalized differentiation, which will be used in the sequel. These notations and facts can be found in [5, 20, 22, 23, 29]. Unless otherwise stated, all spaces under consideration are Asplund spaces whose norms are always denoted by $\|\cdot\|$. The canonical pairing between Z and its dual Z^* is denoted by $\langle\cdot\rangle$. The symbol A^* denotes the adjoint operator of a linear continuous operator A. The opened ball with center $\bar{z}$ and radius ρ is denoted by $B(\bar{z}, \rho)$.

A single-valued mapping $f : Z \to Y$ is said to be *strictly differentiable* at $\bar{z}$ if there is a linear continuous operator $\nabla f(\bar{z}) : Z \to Y$ such that

$$\lim_{z,u\to\bar{z}} \frac{f(z) - f(u) - \langle \nabla f(\bar{z}), z - u\rangle}{\|z - u\|} = 0.$$

Given a multifunction $F : Z \rightrightarrows Z^*$ between a Asplund Z and its dual Z^*, we denote by

$$\operatorname*{Limsup}_{z\to\bar{z}} F(z) : = \Big\{z^* \in Z^* : \exists \text{ sequences } z_n \to \bar{z} \text{ and } z_n^* \overset{w^*}{\to} z^* \\ \text{with } z_n^* \in F(z_n) \text{ for all } n \in \mathbb{N}\Big\}$$

and

$$\operatorname*{Liminf}_{z\to\bar{z}} F(z) : = \Big\{z^* \in Z^* : \forall \text{ sequences } z_n \to \bar{z} \ \exists\, z_n^* \in F(z_n) \text{ with } n \in \mathbb{N} \\ \text{suchthat } z_n^* \overset{w^*}{\to} z^* \text{as } n \to \infty\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer and lower/inner limits* of F as $z \to \bar{z}$ with respect to the norm topology of Z and the weak* topology of Z^*, where $\mathbb{N} := \{1, 2, \ldots\}$.

Let $\varphi : Z \to \bar{R}$ be an extended real-valued function and $\bar{z} \in Z$ be such that $\varphi(\bar{z})$ is finite. For each $\varepsilon \geq 0$, the set

$$\hat{\partial}_\varepsilon \varphi(\bar{z}) := \left\{z^* \in Z^* : \liminf_{z\to\bar{z}} \frac{\varphi(z) - \varphi(\bar{z}) - \langle z^*, z - \bar{z}\rangle}{\|z - \bar{z}\|} \geq -\varepsilon\right\}$$

is called the *ε-Fréchet subdifferential* of φ at $\bar{z}$. A given vector $z^* \in \widehat{\partial}_\varepsilon \varphi(\bar{z})$ is called an *ε-Fréchet subgradient* of φ at $\bar{z}$. The set $\widehat{\partial}\varphi(\bar{z}) = \widehat{\partial}_0\varphi(\bar{z})$ is called the *Fréchet subdifferential* of φ at $\bar{z}$ and the set

$$\partial\varphi(\bar{z}) := \operatorname*{Limsup}_{\substack{z \xrightarrow{\varphi} \bar{z} \\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon \varphi(z) \tag{9}$$

is called the *Mordukhovich subdifferential* of φ at $\bar{z}$, where the notation $z \xrightarrow{\varphi} \bar{z}$ means $z \to \bar{z}$ and $\varphi(z) \to \varphi(\bar{z})$. Hence

$$z^* \in \partial\varphi(\bar{z}) \Longleftrightarrow \text{thereexistsequences } z_k \xrightarrow{\varphi} \bar{z}, \varepsilon_k \to 0^+, \text{ and } z_k^* \in \widehat{\partial}_{\varepsilon_k}\varphi(z_k)$$

such that $z_k^* \xrightarrow{w^*} z^*$. If φ is lower semicontinuous around $\bar{z}$, then we can equivalently put $\varepsilon = 0$ in (9). Moreover, we have $\partial\varphi(\bar{z}) \neq \emptyset$ for every locally Lipschitzian function. It is known that the Mordukhovich subdifferential reduces to the classical Fréchet derivative for strictly differentiable functions and to subdifferential of convex analysis for convex functions.

Suppose that $D \subset Z$, we denote the interior and the closure of D by int D and cl D, respectively. Given a point $\bar{x} \in$ cl D. The *Bouligand tangent cone* (or contingent cone) and the *Clarke tangent cone* to D at $\bar{x}$ are defined by

$$T_B(\bar{z}; D) = \operatorname*{Limsup}_{t\downarrow 0} \frac{D - \bar{z}}{t} = \left\{h \in Z : \exists t_n \to 0^+, \exists h_n \to h, \bar{z} + t_n h_n \in D, \forall n\right\}$$

and

$$T_C(\bar{z}; D) = \operatorname*{Liminf}_{\substack{z \xrightarrow{D} \bar{z} \\ t\downarrow 0}} \frac{D - z}{t} = \left\{h \in Z : \forall t_n \to 0^+, \forall \bar{z}_n \to \bar{z}, \exists h_n \to h, \bar{z}_n + t_n h_n \in D, \forall n\right\},$$

respectively. Note that these cones are closed and $T_C(\bar{z}; D)$ is convex. Moreover,

$$T_C(\bar{z}; D) \subset T_B(\bar{z}; D)$$

and

$$\begin{aligned} T_C(\bar{z}; D) = T_B(\bar{z}; D) = T(\bar{z}; D) = \text{cl}\left(D(\bar{z})\right) &= \text{cl}\left(\text{cone}(D - \bar{z})\right) \\ &= \text{cl}\left\{\lambda(d - \bar{z}) : d \in D, \lambda > 0\right\} \end{aligned}$$

when D is a convex set.

One says that D is *tangentially regular* at $\bar{z}$ if $T_C(\bar{z}; D) = T_B(\bar{z}; D)$. The *negative polar* of the Clarke tangent cone $T_C(\bar{z}; D)$ denoted by $N_C(\bar{z}; D)$ is called the *Clarke normal cone* to D at $\bar{z}$, i.e.,

$$N_C(\bar{z}; D) = T_C(\bar{z}; D)^\circ = \left\{z^* \in Z^* : \langle z^*, z\rangle \leq 0, \forall z \in T_C(\bar{z}; D)\right\}.$$

Let $\varepsilon \geq 0$. The set

$$\widehat{N}_\varepsilon(\bar{z}; D) := \left\{z^* \in Z^* : \limsup_{z \xrightarrow{D} \bar{z}} \frac{\langle z^*, z - \bar{z}\rangle}{\|z - \bar{z}\|} \leq \varepsilon\right\} \tag{10}$$

is called the ε-*Fréchet normal set* to D at $\bar{z}$ and the set

$$N(\bar{z}; D) := \operatorname*{Limsup}_{\substack{z \xrightarrow{D} \bar{z} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(z; D)$$

is called the *Mordukhovich normal cone* to D at $\bar{z}$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{z}; D) = \widehat{N}_0(\bar{z}; D)$ in (10) is a cone called the *Fréchet normal cone* to D at $\bar{z}$.

It is known that (see e.g., [22])

$$\widehat{N}(\bar{z}; D) \subset N(\bar{z}; D) \subset N_C(\bar{z}; D).$$

The set D is said *Fréchet normally regular* at $\bar{z}$ if $\widehat{N}(\bar{z}; D) = N_C(\bar{z}; D)$. We know that the Fréchet normal regularity of a nonempty closed subset D at $\bar{z}$ implies the tangential regularity of D at the corresponding point and if Z is assumed to be a finite dimensional space, then we have the equivalence (see [4]).

It is also known that if Ω is a convex set, then the Mordukhovich normal cone coincides with the Fréchet normal cone, coincides with the Clarke normal cone and coincides with normal cone of convex analysis for convex sets.

The set D is said to be *epi-Lipschitzian* at $\bar{z}$ if there exist a neighborhood U of $\bar{z}$, a number $\lambda > 0$, and a non-empty open set $V \subset Z$ such that

$$z + tv \in D \text{ forall } z \in U \cap D,\ v \in V,\ t \in (0, \lambda).$$

Let $G : W \rightrightarrows Y$ be a set-valued map with the *domain* and the *graph*

$$\text{dom } G := \{w \in W : G(w) \neq \emptyset\}, \quad \text{gph } G := \{(w, y) \in W \times Y : y \in G(w)\}.$$

The symbol G^{-1} denotes the inverse multifunction from Y to W given by

$$G^{-1}(y) := \{w \in W : y \in G(w)\}.$$

Thus,

$$\text{gph } G^{-1} = \{(y, w) \in Y \times W : (w, y) \in \text{gph } G\}.$$

The *Fréchet coderivative* of G at $(\bar{w}, \bar{y}) \in \text{gph } G$ is the multifunction $\hat{D}^* G(\bar{w}, \bar{y}) : Y^* \to W^*$ defined by $\hat{D}^* G(\bar{w}, \bar{y})(y^*) := \{w^* \in W^* : (w^*, -y^*) \in \hat{N}\big((\bar{w}, \bar{y}); \text{gph } G\big)\}$, $y^* \in Y^*$. The *Mordukhovich coderivative* of G at $(\bar{w}, \bar{y}) \in \text{gph } G$ is the multifunction $D^* G(\bar{w}, \bar{y}) : Y^* \to W^*$ defined by $D^* G(\bar{w}, \bar{y})(y^*) := \{w^* \in W^* : (w^*, -y^*) \in N\big((\bar{w}, \bar{y}); \text{gph } G\big)\}$, $y^* \in Y^*$. The *Clarke coderivative* of G at $(\bar{w}, \bar{y}) \in \text{gph } G$ is the multifunction $D^*_C G(\bar{w}, \bar{y}) : Y^* \to W^*$ defined by

$$D^*_C G(\bar{w}, \bar{y})(y^*) := \{w^* \in W^* : (w^*, -y^*) \in N_C\big((\bar{w}, \bar{y}); \text{gph } G\big)\}, \quad y^* \in Y^*.$$

The *mixed coderivative* of G at $(\bar{w}, \bar{y}) \in \operatorname{gph} G$ is the multifunction $D^*_M G(\bar{w}, \bar{y}) : Y^* \to W^*$ defined by

$$D^*_M G(\bar{w}, \bar{y})(y^*) := \Big\{ w_1^* \in W^* : \exists\, \varepsilon_n \downarrow 0, (w_n, y_n) \to (\bar{w}, \bar{y}), w_n^* \xrightarrow{w^*} w_1^*, y_n^* \to y^* \\ \text{with } (w_n^*, -y_n^*) \in \hat{N}_{\varepsilon_n}\big((w_n, y_n); \operatorname{gph} G\big) \Big\}, \quad y^* \in Y^*.$$

It follows from the definitions that

$$\hat{D}^* G(\bar{w}, \bar{y})(y^*) \subset D^*_M G(\bar{w}, \bar{y})(y^*) \subset D^* G(\bar{w}, \bar{y})(y^*) \subset D^*_C G(\bar{w}, \bar{y})(y^*), \forall y^* \in Y^*.$$

Suppose that $E \subset Y$ is a pointed closed convex cone, i.e., $E \cap (-E) = \{0\}$ and E induces a partial order $\preceq_E$ on Y, i.e.,

$$y \preceq_E y' \Leftrightarrow y' - y \in E, \ \forall y, y' \in Y.$$

A single-valued mapping $l : V \subset W \to Y$ is said to be *locally upper Lipschitzian* (respectively, *locally Lipschitzian*) at $\bar{w} \in V$ if there are numbers $\eta > 0$ and $\ell \geq 0$ such that

$$\|l(w) - l(\bar{w})\| \leq \ell \|w - \bar{w}\|, \text{ forall } w \in B_\eta(\bar{w}) \cap V$$
$$\big(\text{respectively, } \|l(w) - l(w')\| \leq \ell \|w - w'\|, \text{ forall } w, w' \in B_\eta(\bar{w}) \cap V\big).$$

We say that a multifunction $L : W \rightrightarrows Y$ admits a *local upper Lipschitzian selection* at $(\bar{w}, \bar{y}) \in \operatorname{gph} L$ if there is a single-valued mapping $l : \operatorname{dom} L \to Y$ which is locally upper Lipschitzian at $\bar{w}$ satisfying $l(\bar{w}) = \bar{y}$ and $l(w) \in L(w)$ for all $w \in \operatorname{dom} L$ in a neighborhood of $\bar{w}$.

3 Sensitivity Analysis in Multi-objective Programming Problems

In this section, we suppose that X, W and Z are Asplund spaces with the dual spaces X^*, W^* and Z^*, respectively. Assume that $g : W \times Z \to X$ is a continuous mapping. Let $f : W \times Z \to \mathbb{R}^s$ be a vector function and Ω be a closed and convex set in Z. For each $w \in W$, we put

$$G(w) := \big\{ z \in Z : \ g(w, z) = 0 \big\}.$$

Consider the problem

$$\operatorname{Min}_{\mathbb{R}^s_+} \ f(w, z), \quad \text{subject to } \ z \in G(w) \cap \Omega. \tag{11}$$

Let $\tilde{F} : W \rightrightarrows \mathbb{R}^s$ be the multifunction given by

$$\tilde{F}(w) = (f \diamond G_\Omega)(w) := \{ f(w, z) : z \in G_\Omega(w) \},$$

where $G_\Omega(w) = G(w) \cap \Omega,\ \forall w \in W$. We put

$$\tilde{\mathcal{F}}(w) = \mathrm{Min}_{\mathbb{R}^s_+} \tilde{F}(w), \quad w \in W$$

and call $\tilde{\mathcal{F}} : W \rightrightarrows \mathbb{R}^s$ the *efficient point multifunction* or *the frontier map* of the problem (11). The point $\bar{z} \in G(\bar{w}) \cap \Omega$ such that $f(\bar{w}, \bar{z}) \in \tilde{\mathcal{F}}(\bar{w})$ is called *local weak Pareto solution* of the problem (11) at $\bar{w}$.

Thus, $G_\Omega : W \rightrightarrows Z$ be a multifunction with the domain and the graph

$$\begin{aligned} &\mathrm{dom}\ G_\Omega = \{w \in W : G(w) \cap \Omega \neq \emptyset\},\\ &\mathrm{gph}\ G_\Omega = \{(w, y) \in W \times Y : y \in G(w) \cap \Omega\}. \end{aligned}$$

This section is allocated to establish formulas for computing the Clarke coderivative of the efficient point multifunction $\tilde{\mathcal{F}}$. We first establish a formula for exact computing the Clarke coderivative of the constraint function G_Ω.

Proposition 3.1 *Suppose that Ω is locally closed around $\bar{z}$, epi-Lipschitzian at $\bar{z}$. Assume further that the function g is differentiable around $(\bar{w}, \bar{z})$, ∇g is continuous at $(\bar{w}, \bar{z})$, $\nabla_z g(\bar{w}, \bar{z})$ or $\nabla_w g(\bar{w}, \bar{z})$ is surjective, and the following regularity conditions is satisfied*

$$\Big\{(w, z) \in W \times Z : \nabla g(\bar{w}, \bar{z})(w, z) = 0\Big\} \cap \Big[W \times \mathrm{int}\, T(\bar{z}, \Omega)\Big] \neq \emptyset. \tag{12}$$

Then for each $(w^*, z^*) \in W^* \times Z^*$,

$$D^*_C G_\Omega(\bar{w}, \bar{z})(w^*, z^*) = - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \big[\nabla_w g(\bar{w}, \bar{z})^* \big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(z^* + z_1^*)\big)\big].$$

Proof Put $B = \mathrm{gph}\ G,\ D = W \times \Omega$. We first prove that

$$N_C\big((\bar{w}, \bar{z}); B\big) = \hat{N}\big((\bar{w}, \bar{z}); B\big) = \Big\{\big(\nabla_w g(\bar{w}, \bar{z})^* x^*, \nabla_z g(\bar{w}, \bar{z})^* x^*\big) : x^* \in X^*\Big\}. \tag{13}$$

Note that B can be represented in the form

$$B = \{(w, z) \in W \times Z : g(w, z) = 0\} = g^{-1}(0).$$

From $\nabla g(\bar{w}, \bar{z})(w, z) = \nabla_w g(\bar{w}, \bar{z})w + \nabla_z g(\bar{w}, \bar{z})z$ and $\nabla_z g(\bar{w}, \bar{z})$ or $\nabla_w g(\bar{w}, \bar{z})$ is surjective, we get that $\nabla g(\bar{w}, \bar{z})$ is also surjective. By [22, Theorem 1.14 and Corollary 1.15], we get

$$\begin{aligned} \hat{N}\big((\bar{w}, \bar{z}); D\big) &= \hat{N}\big((\bar{w}, \bar{z}); g^{-1}(0)\big)\\ &= \nabla g(\bar{w}, \bar{z})^* \hat{N}\big(g(\bar{w}, \bar{z}); \{0\}\big) = \nabla g(\bar{w}, \bar{z})^*(X^*). \end{aligned}$$

Since the strictly differentiability of function g at $(\bar{w}, \bar{z})$ and [39, Lemma 3.5], we have

$$\begin{aligned} N_C\big((\bar{w}, \bar{z}); D\big) &= N_C\big((\bar{w}, \bar{z}); g^{-1}(0)\big) \\ &= \nabla g(\bar{w}, \bar{z})^* N_C\big(g(\bar{w}, \bar{z}); \{0\}\big) = \nabla g(\bar{w}, \bar{z})^*(X^*). \end{aligned}$$

Thus, we obtain (13). We now prove that

$$N_C\big((\bar{w}, \bar{z}); B \cap D\big) = \{0\} \times N(\bar{z}; \Omega) + N_C\big((\bar{w}, \bar{z}); B\big). \tag{14}$$

Since Ω is epi-Lipschitzian at $\bar{z}$, we have that D is also epi-Lipschitzian at $(\bar{w}, \bar{z})$. By (13), we get

$$\begin{aligned} T_C((\bar{w}, \bar{z}); B) &= \hat{T}((\bar{w}, \bar{z}); B) \\ &= \Big\{(w, z) \in W \times Z : \langle (w^*, z^*), (w, z)\rangle \le 0,\ \forall (w^*, z^*) \in N_C\big((\bar{w}, \bar{z}); B\big)\Big\} \\ &= \Big\{(w, z) \in W \times Z : \langle (w, z), \nabla g(\bar{w}, \bar{z})^* x^*\rangle \le 0,\ \forall x^* \in X^*\Big\} \\ &= \Big\{(w, z) \in W \times Z : \langle \nabla g(\bar{w}, \bar{z})(w, z), x^*\rangle \le 0,\ \forall x^* \in X^*\Big\} \\ &= \Big\{(w, z) \in W \times Z : \nabla g(\bar{w}, \bar{z})(w, z) = 0\Big\}. \end{aligned}$$

Combining this and (12), we have $T_C((\bar{w}, \bar{z}); B) \cap \operatorname{int} T_C((\bar{w}, \bar{z}); D) \neq \emptyset$. Note that B and D are Fréchet normally regular at $(\bar{w}, \bar{z})$. By [4, Theorem 6.2], B and D are also Fréchet tangentially regular at $(\bar{w}, \bar{z})$. By [28, Corollary 3], we obtain that

$$\begin{aligned} N_C\big((\bar{w}, \bar{z}); B \cap D\big) = N_C((\bar{w}, \bar{z}); D) + N_C\big((\bar{w}, \bar{z}); B\big) &= \{0\} \times N(\bar{z}; \Omega) \\ &\quad + N_C\big((\bar{w}, \bar{z}); B\big), \end{aligned}$$

this is formula (14). Since the definition of the Clarke coderivative, we get

$$\begin{aligned} D_C^* G_\Omega(\bar{w}, \bar{z})(z^*) &= \{w_1^* \in W^* : (w_1^*, -z^*) \in N_C\big((\bar{w}, \bar{z}); \operatorname{gph} G_\Omega\big)\} \\ &= \{w_1^* \in W^* : (w_1^*, -z^*) \in N_C\big((\bar{w}, \bar{z}); C \cap D\big)\}. \end{aligned}$$

From (14), we have

$$D_C^* G_\Omega(\bar{w}, \bar{z})(z^*) = \{w^* \in W^* : (w^*, -z^*) \in \{0\} \times N_C(\bar{z}; \Omega) + N_C\big((\bar{w}, \bar{z}); B\big)\}.$$

We note that $(w^*, -z^*) \in \{0\} \times N_C(\bar{z}; \Omega) + N_C\big((\bar{w}, \bar{z}); B\big)$ if and only if there exists $z_1^* \in N_C(\bar{z}; \Omega) = \hat{N}(\bar{z}; \Omega)$ such that $(w^*, -z^* - z_1^*) \in N_C\big((\bar{w}, \bar{z}); B\big)$. Since (13), there exists $x^* \in X^*$ such that $w^* = \nabla_w g(\bar{w}, \bar{z})^*(x^*)$ and $-z_1^* - z^* = \nabla_z g(\bar{w}, \bar{z})^*(x^*)$. This follows that $w^* = -\nabla_w g(\bar{w}, \bar{z})^*(-x^*)$ and $z_1^* + z^* = \nabla_z g(\bar{w}, \bar{z})^*(-x^*)$. So

$$w^* \in -\nabla_w g(\bar{w}, \bar{z})^*\big[(\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(z_1^* + z^*)\big].$$

Thus,

$$D_C^* G_\Omega(\bar{w}, \bar{z})(z^*) = - \bigcup_{z_1^* \in N(\bar{z};\Omega)} \nabla_w g(\bar{w}, \bar{z})^* \big[(\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(z_1^* + z^*)\big].$$

The proof of the proposition is complete. □

Note that $T(\bar{z}; \Omega) = Z$ where $\bar{z} \in \operatorname{int} \Omega$. and

$$\Big\{(w, z) \in W \times Z : \nabla g(\bar{w}, \bar{z})(w, z) = 0\Big\} \neq \emptyset,$$

for all $(w, z) \in W \times Z$. So, the condition (12) is satisfied if $\bar{z} \in \operatorname{int} \Omega$. Moreover, since $\bar{z} \in \operatorname{int} \Omega$, there exists a ball $B(\bar{z}, \epsilon)$ with radius ϵ, center $\bar{z}$ such that $B(\bar{z}, \epsilon) \subset \Omega$. Choose $U = B\Big(\bar{z}, \frac{\epsilon}{2}\Big),\ V = B\Big(0, \frac{\epsilon}{2}\Big)$ and $\lambda = 1$, we have

$$\|z_1 + tz_2 - \bar{z}\| \leq \|z_1 - \bar{z}\| + t\|z_2\| < \frac{\epsilon}{2} + t\frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $z_1 \in U,\ z_2 \in V$ and $t \in (0, \lambda)$. So, $z_1 + tz_2 \in B \subset \Omega$ for all $z_1 \in U,\ z_2 \in V$ and $t \in (0, \lambda)$. This means that Ω epi-Lipschitzian at $\bar{z}$.

The *uniformly positive polar* to cone $K \subset \mathbb{R}^s$ (see [10]) is defined by

$$K_{up}^* := \{y^* \in \mathbb{R}^s : \exists \beta > 0,\ \langle y^*, k\rangle \geq \beta |k|,\ \ \forall k \in K\}.$$

We estimate the Clarke coderivatives of the sum of a multifunction $\tilde{\mathcal{F}}$ and cone $K = \mathbb{R}^s_+$ by the following proposition.

Proposition 3.2 *Let $\hat{\mathcal{F}} : W \times \mathbb{R}^s \to \mathbb{R}^s$ be a multifunction defined by $\hat{\mathcal{F}}(w, y) = \tilde{\mathcal{F}}(w) \cap (y - \mathbb{R}^s_+)$.*

(i) If $\tilde{\mathcal{F}} + \mathbb{R}^s_+$ is tangentially regular at $(\bar{w}, \bar{y})$, then one has

$$D_C^*(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*) \subset D_C^*\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*),\ \ y^* \in \mathbb{R}^s;$$

(ii) If $\tilde{\mathcal{F}}$ is Fréchet normally regular at $(\bar{w}, \bar{y})$ and $\hat{\mathcal{F}}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$, then one has

$$D_C^*(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*) \supset D_C^*\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*),\ \ y^* \in K^*_{\mathrm{up}} = \operatorname{int} \mathbb{R}^s_+,$$

where $K = \mathbb{R}^s_+$.

Proof We first prove assertion (i). It is easy to see that gph $\tilde{\mathcal{F}} \subset$ gph $(\tilde{\mathcal{F}} + \mathbb{R}^s_+)$. By the assumption of proposition and the monotonicity property of the Bouligand tangent cone, we get

$$T_C\big((\bar{w}, \bar{y}); \operatorname{gph} \tilde{\mathcal{F}}\big) \subset T_B\big((\bar{w}, \bar{y}); \operatorname{gph} \tilde{\mathcal{F}}\big) \subset T_B\big((\bar{w}, \bar{y}); \operatorname{gph} (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big)$$

$$= T_C\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big).$$

So,

$$N_C\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big) \subset N_C\big((\bar{w}, \bar{y}); \text{gph } \tilde{\mathcal{F}}\big).$$

Since the definition of the Clarke coderivative, we have assertion (i) of proposition. To prove assertion (ii), we first note that

$$K^*_{\text{up}} = \text{int } \mathbb{R}^s_+,$$

where $K = \mathbb{R}^s_+$. We now take $y^* \in K^*_{\text{up}}$ and $w^* \in D^*_C\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*)$. Assume for contradiction that $w^* \notin D^*_C(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*)$. Since the definition of the Clarke coderivative, $(w^*, -y^*) \notin N_C\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big)$. Note that

$$\hat{N}\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big) \subset N_C\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big).$$

So, $(w^*, -y^*) \notin \hat{N}\big((\bar{w}, \bar{y}); \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)\big)$. By the definition of the Fréchet normal cone, there is $(w_n, y_n) \to (\bar{w}, \bar{y})$ with $y_n \in \tilde{\mathcal{F}}(w_n) + \mathbb{R}^s_+$ such that

$$\limsup_{n\to\infty} \frac{\langle (w^*, -y^*), (w_n, y_n) - (\bar{w}, \bar{y})\rangle}{\|(w_n, y_n) - (\bar{w}, \bar{y})\|} > 0. \tag{15}$$

Note that dom $\hat{\mathcal{F}} = \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)$. From $\hat{\mathcal{F}}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$, there are $l > 0$ and $U \times V$ is a neighborhood of $(\bar{w}, \bar{y})$ such that for each $(u, y) \in (U \times V) \cap \text{gph } (\tilde{\mathcal{F}} + \mathbb{R}^s_+)$, we can find $y' \in \hat{\mathcal{F}}(w, y)$ satisfying $\|y' - \bar{y}\| \le l\|(w, y) - (\bar{w}, \bar{y})\|$. Since $(w_n, y_n) \to (\bar{w}, \bar{y})$, there is $n_0 \in \mathbb{N}$ such that $(w_n, y_n) \in U \times V$, for all $n > n_0$. Thus, for each $n > n_0$, there exists $y'_n \in \hat{\mathcal{F}}(w_n, y_n) = \tilde{\mathcal{F}}(w_n) \cap (y_n - \mathbb{R}^s_+)$ such that $\|y'_n - \bar{y}\| \le l\|(w_n, y_n) - (\bar{w}, \bar{y})\|$. So, for each $n > n_0$, there are $y'_n \in \tilde{\mathcal{F}}(w_n)$ and $k_n \in \mathbb{R}^s_+$ such that $y'_n = y_n - k_n$. This is equivalent to that there is $(w_n, y'_n) \xrightarrow{\text{gph } \tilde{\mathcal{F}}} (\bar{w}, \bar{y})$ such that

$$\begin{aligned}
\|(w_n, y'_n) - (\bar{w}, \bar{y})\| &= \|(w_n - \bar{w}, y'_n - \bar{y})\| = \|w_n - \bar{w}\| + \|y'_n - \bar{y}\| \\
&\le \|w_n - \bar{w}\| + \|y_n - \bar{y}\| + \|y'_n - \bar{y}\| = \|(w_n - \bar{w}, y_n - \bar{y})\| \\
&\quad + \|y'_n - \bar{y}\| \\
&\le \|(w_n - \bar{w}, y_n - \bar{y})\| + l\|(w_n - \bar{w}, y_n - \bar{y})\| \\
&= (l + 1)\|(w_n - \bar{w}, y_n - \bar{y})\|.
\end{aligned}$$

Since $y^* \in K^+_{\text{up}} = \text{int } \mathbb{R}^s_+$ and $k_n \in \mathbb{R}^s_+$, we have $\langle y^*, k_n\rangle \ge 0,\ \forall n$. So, for each $n \ge n_0$, we get

$$\langle (w^*, -y^*), (w_n, y'_n) - (\bar{w}, \bar{y})\rangle = \langle (w^*, -y^*), (w_n - \bar{w}, y_n - k_n - \bar{y})\rangle$$

$$= \langle (w^*, -y^*), (w_n - \bar{w}, y_n - \bar{y}) \rangle + \langle y^*, k_n \rangle$$
$$\geq \langle (w^*, -y^*), (w_n - \bar{w}, y_n - \bar{y}) \rangle.$$

Combining this and (15), we have

$$\limsup_{n\to\infty} \frac{\langle (w^*, -y^*), (w_n, y_n') - (\bar{w}, \bar{y}) \rangle}{\|(w_n, y_n') - (\bar{w}, \bar{y})\|} \geq \limsup_{n\to\infty} \frac{\langle (w^*, -y^*), (w_n, y_n) - (\bar{w}, \bar{y}) \rangle}{\|(w_n, y_n') - (\bar{w}, \bar{y})\|}$$
$$\geq \limsup_{n\to\infty} \frac{\langle (w^*, -y^*), (w_n, y_n) - (\bar{w}, \bar{y}) \rangle}{(l+1)\|(w_n, y_n) - (\bar{w}, \bar{y})\|} > 0.$$

So,

$$\limsup_{(w,y)\xrightarrow{\text{gph } \tilde{\mathcal{F}}}(\bar{w},\bar{y})} \frac{\langle (w^*, -y^*), (w, y) - (\bar{w}, \bar{y}) \rangle}{\|(w, y) - (\bar{w}, \bar{y})\|} > 0.$$

This is equivalent to $(w^*, -y^*) \notin \hat{N}\big((\bar{w}, \bar{y}); \text{gph } \tilde{\mathcal{F}}\big)$. Hence, $w^* \notin D^*_C \tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*)$, a contradiction. The proof of proposition is complete. □

Next, we establish outer/inner estimates for the Clarke coderivative of $\tilde{F}$.

Proposition 3.3 *Let Ω be locally closed around $\bar{z}$, epi-Lipschitzian at $\bar{z}$ and let $\tilde{G}_\Omega : W \times \mathbb{R}^s \to Z$ be a multifunction defined by $\tilde{G}_\Omega(w, y) = \{z \in G_\Omega(w) : y = f(w, z)\}$. Suppose that the function g is differentiable around $(\bar{w}, \bar{z})$, ∇g is continuous at $(\bar{w}, \bar{z})$, $\nabla_z g(\bar{w}, \bar{z})$ or $\nabla_w g(\bar{w}, \bar{z})$ is surjective, and the following regularity conditions is satisfied*

$$\Big\{(w, z) \in W \times Z : \nabla g(\bar{w}, \bar{z})(w, z) = 0\Big\} \cap \Big[W \times \text{int } T(\bar{z}, \Omega)\Big] \neq \emptyset.$$

Assume further that $\bar{w} \in W$, $\bar{y} \in \tilde{F}(\bar{w})$ and $\bar{z} \in G_\Omega(\bar{w}) = G(\bar{w}) \cap \Omega$ satisfying $(\bar{w}, \bar{z}) \in f^{-1}(\bar{y})$, the function f is Fréchet differentiable at $(\bar{w}, \bar{z})$ with the derivative $\nabla f(\bar{w}, \bar{z}) = \big(\nabla_w f(\bar{w}, \bar{z}), \nabla_z f(\bar{w}, \bar{z})\big)$.

(i) If $\tilde{F}$ is tangentially regular at $(\bar{w}, \bar{y})$, then one has

$$D^*_C \tilde{F}(\bar{w}, \bar{y})(y^*) \subset \nabla_w f(\bar{w}, \bar{z})^*(y^*)$$
$$- \bigcup_{z_1^* \in \hat{N}(\bar{z},\Omega)} \big[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\big],$$

for all $y^* \in \mathbb{R}^s$;

(ii) If G_Ω is Fréchet normally regular at $(\bar{w}, \bar{z})$ and $\tilde{G}_\Omega$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{z})$, then

$$D^*_C \tilde{F}(\bar{w}, \bar{y})(y^*) \supset \nabla_w f(\bar{w}, \bar{z})^*(y^*)$$

$$-\bigcup_{z_1^*\in\hat{N}(\bar{z},\Omega)}\Big[\nabla_w g(\bar{w},\bar{z})^*\big((\nabla_z g(\bar{w},\bar{z})^*)^{-1}(\nabla_z f(\bar{w},\bar{z})^*(y^*)+z_1^*)\big)\Big],$$

for all $y^*\in\mathbb{R}^s$.

Proof To prove assertion (i), we first prove that

$$\Big\{(w,\nabla f(\bar{w},\bar{z})(w,z)) : (w,z)\in T_C\big((\bar{w},\bar{z});\text{gph } G_\Omega\big)\Big\}\subset T_C\big((\bar{w},\bar{y});\text{gph } \tilde{F}\big),\quad \forall w\in W. \tag{16}$$

For each $w\in W$, put $(w,z)\in T_C\big((\bar{w},\bar{z});\text{gph } G_\Omega\big)\subset T_B\big((\bar{w},\bar{z});\text{gph } G_\Omega\big)$. Then, there are sequences $\{t_n\}\subset(0,+\infty)$, $t_n\to 0$ and $\{(w_n,z_n)\}\subset W\times Z$, $(w_n,z_n)\to(w,z)$ with $\bar{z}+t_nz_n\in G_\Omega(\bar{w}+t_nw_n)$ for all $n\in\mathbb{N}$. We get

$$f(\bar{w}+t_nw_n,\bar{z}+t_nz_n)\in\tilde{F}(\bar{w}+t_nw_n),\ \forall n.$$

This is equivalent to

$$\bar{y}+t_n\frac{f(\bar{w}+t_nw_n,\bar{z}+t_nz_n)-f(\bar{w},\bar{z})}{t_n}\in\tilde{F}(\bar{w}+t_nw_n),\ \forall n.$$

By the Fréchet differentiable property of f at $(\bar{w},\bar{z})$, we have

$$\lim_{n\to\infty}\frac{f(\bar{w}+t_nw_n,\bar{z}+t_nz_n)-f(\bar{w},\bar{z})}{t_n}=\nabla f(\bar{w},\bar{z})(w,z).$$

This implies that

$$\big(w,\nabla f(\bar{w},\bar{z})(w,z)\big)\in T_B\big((\bar{w},\bar{y});\text{gph } \tilde{F}\big)=T_C\big((\bar{w},\bar{y});\text{gph } \tilde{F}\big).$$

Thus, (16) is proved. For each $y^*\in\mathbb{R}^s$, we now take any $w^*\in D_C^*\tilde{F}(\bar{w},\bar{y})(y^*)$. By the definition of the Clarke coderivative, we get

$$(w^*,-y^*)\in N_C\big((\bar{w},\bar{y});\text{gph } \tilde{F}\big). \tag{17}$$

We now prove that

$$N_C\big((\bar{w},\bar{y});\text{gph } \tilde{F}\big)\subset\Big\{\big(\nabla_w f(\bar{w},\bar{z})^*(y^*)+u^*,-y^*\big) : \big(u^*,-\nabla_z f(\bar{w},\bar{z})^*(y^*)\big)\in N_C\big((\bar{w},\bar{z});\text{gph } G_\Omega\big)\Big\}. \tag{18}$$

Since (16), inclusion (18) is proved if we can show

$$\langle\big(\nabla_w f(\bar{w},\bar{z})^*(y^*)+u^*,-y^*\big),\big(w,\nabla f(\bar{w},\bar{z})(w,z)\big)\rangle\le 0,$$

for all $(w,z) \in T_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big)$ and for all $\big(u^*, -\nabla_z f(\bar{w},\bar{z})^*(y^*)\big) \in N_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big)$. This is always true, because for each $(w,z) \in T_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big)$ and for all

$$\big(u^*, -\nabla_z f(\bar{w},\bar{z})^*(y^*)\big) \in N_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big),$$

we have

$$\begin{aligned}&\Big\langle\big(\nabla_w f(\bar{w},\bar{z})^*(y^*) + u^*, -y^*\big), \big(w, \nabla f(\bar{w},\bar{z})(w,z)\big)\Big\rangle \\ &\quad = \Big\langle\big(\nabla_w f(\bar{w},\bar{z})^*(y^*) + u^*, -y^*\big), \big(w, \nabla_w f(\bar{w},\bar{z})w + \nabla_z f(\bar{w},\bar{z})z\big)\Big\rangle \\ &\quad = \nabla_w f(\bar{w},\bar{z})^* y^*(w) + u^*(w) - \nabla_w f(\bar{w},\bar{z})^* y^*(w) - \nabla_z f(\bar{w},\bar{z})^* y^*(z) \\ &\quad = u^*(w) - \nabla_z f(\bar{w},\bar{z})^* y^*(z) \le 0.\end{aligned}$$

Combining (17) and (18), there exists $\big(u^*, -\nabla_z f(\bar{w},\bar{z})^*(y^*)\big) \in N_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big)$ such that $(w^*, -y^*) = \big(\nabla_w f(\bar{w},\bar{z})^*(y^*) + u^*, -y^*\big)$. This implies that

$$\big(w^* - \nabla_w f(\bar{w},\bar{z})^*(y^*), -\nabla_z f(\bar{w},\bar{z})^*(y^*)\big) \in N_C\big((\bar{w},\bar{z}); \text{gph } G_\Omega\big).$$

Using the definition of the Clarke coderivative, we get

$$\begin{aligned}&w^* \\ &-\nabla_w f(\bar{w},\bar{z})^*(y^*) \in D_C^* G_\Omega(\bar{w},\bar{z})\big(\nabla_z f(\bar{w},\bar{z})^*(y^*)\big).\end{aligned}$$

So, $w^* \in \nabla_w f(\bar{w},\bar{z})^*(y^*) + D_C G_\Omega(\bar{w},\bar{z})\big(\nabla_z f(\bar{w},\bar{z})^*(y^*)\big)$. By Proposition 3.1,

$$\begin{aligned}&w^* \in \nabla_w f(\bar{w},\bar{z})^*(y^*) \\ &- \bigcup_{z_1^* \in \hat{N}(\bar{z},\Omega)} \big[\nabla_w g(\bar{w},\bar{z})^*\big((\nabla_z g(\bar{w},\bar{z})^*)^{-1}(\nabla_z f(\bar{w},\bar{z})^*(y^*) + z_1^*)\big)\big].\end{aligned}$$

Thus, assertion (i) is proved. We now prove assertion (ii). Take any $w^* \notin D_C^* \tilde{F}(\bar{w},\bar{y})(y^*)$, we will prove that

$$\begin{aligned}&w^* \notin \nabla_w f(\bar{w},\bar{z})^*(y^*) \\ &- \bigcup_{z_1^* \in \hat{N}(\bar{z},\Omega)} \big[\nabla_w g(\bar{w},\bar{z})^*\big((\nabla_z g(\bar{w},\bar{z})^*)^{-1}(\nabla_z f(\bar{w},\bar{z})^*(y^*) + z_1^*)\big)\big].\end{aligned}$$

Since Proposition 3.1, we need to prove

$$w^* \notin \nabla_w f(\bar{w},\bar{z})^*(y^*) + D_C^* G_\Omega(\bar{w},\bar{z})\big(\nabla_z f(\bar{w},\bar{z})^*(y^*)\big).$$

From $w^* \notin D_C^* \tilde{F}(\bar{w}, \bar{y})(y^*)$, we have $(w^*, -y^*) \notin N_C\big((\bar{w}, \bar{y}); \text{gph } \tilde{F}\big)$. So $(w^*, -y^*) \notin \hat{N}\big((\bar{w}, \bar{y}); \text{gph } \tilde{F}\big)$. By the definition of Fréchet normal cone,

$$\limsup_{(w,y) \xrightarrow{\text{gph } \tilde{F}} (\bar{w}, \bar{y})} \frac{\langle (w^*, -y^*), (w, y) - (\bar{w}, \bar{y}) \rangle}{\|(w, y) - (\bar{w}, \bar{y})\|} > 0.$$

So, there is $\{(w_n, y_n)\} \subset \text{gph } \tilde{F}$ and $\alpha > 0$ such that $(w_n, y_n) \to (\bar{w}, \bar{y})$ as $n \to \infty$, with

$$\langle w^*, w_n - \bar{w} \rangle \geq \langle y^*, y_n - \bar{y} \rangle + \alpha\big(\|w_n - \bar{w}\| + \|y_n - \bar{y}\|\big) \tag{19}$$

for all n sufficiently large. From $\hat{G}_\Omega$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{z})$, there exists $l : \text{dom } \hat{G}_\Omega \to Z$ satisfying $l(\bar{w}, \bar{y}) = \bar{z},\ l(w, y) \in \hat{G}_\Omega(w, y)$ for all $(w, y) \in \text{dom } \hat{G}_\Omega$ sufficiently close to $(\bar{w}, \bar{y})$ and that l is local upper Lipschitzian at $(\bar{w}, \bar{y})$. So, there is $\ell > 0$ such that

$$\|z_n - \bar{z}\| \leq \ell\big(\|w_n - \bar{w}\| + \|y_n - \bar{y}\|\big) \tag{20}$$

for all n sufficiently large, where $z_n = l(w_n, y_n) \in \hat{G}_\Omega(w_n, y_n)$. From $z_n \in \hat{G}_\Omega(w_n, y_n)$, we have $z_n \in G_\Omega(w_n),\ y_n = f(w_n, z_n)$. Combining this and (19), we get

$$\begin{aligned}
\langle w^*, w_n - \bar{w} \rangle &\geq \langle y^*, f(w_n, z_n) - f(\bar{w}, \bar{z}) \rangle + \alpha\big(\|w_n - \bar{w}\| + \|f(w_n, z_n) - f(\bar{w}, \bar{z})\|\big) \\
&= \langle y^*, \nabla f(\bar{w}, \bar{z})(w_n - \bar{w}, z_n - \bar{z}) \rangle + o(\|w_n - \bar{w}\| + \|z_n - \bar{z}\|) \\
&\qquad + \alpha\big(\|w_n - \bar{w}\| + \|f(w_n, z_n) - f(\bar{w}, \bar{z})\|\big) \\
&= \langle \nabla f(\bar{w}, \bar{z})^*(y^*), (w_n - \bar{w}, z_n - \bar{z}) \rangle + o(\|w_n - \bar{w}\| + \|z_n - \bar{z}\|) \\
&\qquad + \alpha\big(\|w_n - \bar{w}\| + \|f(w_n, z_n) - f(\bar{w}, \bar{z})\|\big).
\end{aligned} \tag{21}$$

Since (20),

$$\alpha\|f(w_n, z_n) - f(\bar{w}, \bar{z})\| \geq \frac{\alpha}{2}\|f(w_n, z_n) - f(\bar{w}, \bar{z})\| \geq \frac{\alpha}{2\ell}\|z_n - \bar{z}\| - \frac{\alpha}{2}\|w_n - \bar{w}\|.$$

Combining this and (21), we have

$$\begin{aligned}
\langle w^*, w_n - \bar{w} \rangle &\geq \langle \nabla f(\bar{w}, \bar{z})^*(y^*), (w_n - \bar{w}, z_n - \bar{z}) \rangle + o(\|w_n - \bar{w}\| + \|z_n - \bar{z}\|) \\
&\qquad + \frac{\alpha}{2}\|w_n - \bar{w}\| + \frac{\alpha}{2\ell}\|z_n - \bar{z}\|\big) \\
&\geq \langle \nabla f(\bar{w}, \bar{z})^*(y^*), (w_n - \bar{w}, z_n - \bar{z}) \rangle + o(\|w_n - \bar{w}\| + \|z_n - \bar{z}\|) \\
&\qquad + \hat{\alpha}\big(\|w_n - \bar{w}\| + \|z_n - \bar{z}\|\big),
\end{aligned}$$

with $\hat{\alpha} = \min\{\frac{\alpha}{2}, \frac{\alpha}{2\ell}\}$. Thus,

$$\limsup_{(w,z) \xrightarrow{\text{gph } G_\Omega} (\bar{w},\bar{z})} \frac{\langle (w^* - \nabla_w f(\bar{w}, \bar{z})^*(y^*), w - \bar{w}\rangle - \langle \nabla_z f(\bar{w}, \bar{z})^*(y^*), z - \bar{z}\rangle}{\|w - \bar{w}\| + \|z - \bar{z}\|} \geq \hat{\alpha},$$

which means that $\big(w^* - \nabla_w f(\bar{w}, \bar{z})^*(y^*), -\nabla_z f(\bar{w}, \bar{z})^*(y^*)\big) \notin \hat{N}\big((\bar{w}, \bar{z}); \text{gph } G_\Omega\big)$. By the Fréchet normal regularity of G_Ω at $(\bar{w}, \bar{z})$, $w^* - \nabla_w f(\bar{w}, \bar{z})^*(y^*) \notin D_C^* G_\Omega(\bar{w}, \bar{z})\big(\nabla_z f(\bar{w}, \bar{z})^*(y^*)\big)$. By Proposition 3.1, we obtain

$$\begin{aligned} &w^* - \nabla_w f(\bar{w}, \bar{z})^*(y^*) \notin \\ &- \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \big[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\big]. \end{aligned}$$

Thus, assertion (ii) is proved. □

We say that the *domination property* holds for a set-valued map $\tilde{F} : W \rightrightarrows \mathbb{R}^s$ around $\bar{w} \in W$, if there exists a neighborhood V of $\bar{w}$ such that $\tilde{F}(w) \subset \text{Min}_{\mathbb{R}^s_+} \tilde{F}(w) + \mathbb{R}^s_+, \ \forall w \in V$. The reader is referred to [21] for discussions and examples.

We consider the multifunctions $\hat{\mathcal{F}}, \tilde{G}_\Omega$ which are defined in Proposition 3.2 and 3.3, respectively. The following theorem gives inner and outer estimates on the Clarke coderivative of the extremum multifunction $\tilde{\mathcal{F}}$, which is the main result of this section.

Theorem 3.1 *Let Ω be locally closed around $\bar{z}$, epi-Lipschitzian at $\bar{z}$ and $\bar{w} \in W$, $\bar{z} \in G_\Omega(\bar{w}) = G(\bar{w}) \cap \Omega$ be such that $\bar{y} = f(\bar{w}, \bar{z}) \in \tilde{\mathcal{F}}(\bar{w})$. Suppose that the function g is differentiable around $(\bar{w}, \bar{z})$, ∇g is continuous at $(\bar{w}, \bar{z})$, $\nabla_z g(\bar{w}, \bar{z})$ or $\nabla_w g(\bar{w}, \bar{z})$ is surjective, and the following regularity conditions is satisfied*

$$\Big\{(w, z) \in W \times Z : \nabla g(\bar{w}, \bar{z})(w, z) = 0\Big\} \cap \Big[W \times \text{int } T(\bar{z}, \Omega)\Big] \neq \emptyset. \qquad (22)$$

Assume further that the function f is Fréchet differentiable at $(\bar{w}, \bar{z})$ with the derivative $\nabla f(\bar{w}, \bar{z}) = \big(\nabla_w f(\bar{w}, \bar{z}), \nabla_z f(\bar{w}, \bar{z})\big)$, the domination property holds for $\tilde{F}$ around $\bar{w}$ and $\hat{\mathcal{F}}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$.

(i) Suppose that $\tilde{F}$ and $\tilde{F} + \mathbb{R}^s_+$ are tangentially regular at $(\bar{w}, \bar{y})$. If $\tilde{\mathcal{F}}$ is Fréchet normally regular at $(\bar{w}, \bar{y})$, then one has

$$\begin{aligned} D_C^* \tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) \subset\ & \nabla_w f(\bar{w}, \bar{z})^*(y^*) \\ & - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \big[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\big], \end{aligned}$$

for all $y^* \in \text{int } \mathbb{R}^s_+$;

(ii) Suppose that $\tilde{\mathcal{F}} + \mathbb{R}^s_+$ is tangentially regular at $(\bar{w}, \bar{y})$ and $\tilde{F}$ is Fréchet normally regular at this point. If G_Ω is Fréchet normally regular at $(\bar{w}, \bar{z})$ and $\tilde{G}_\Omega$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{z})$, then

$$D^*_C\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) \supset \nabla_w f(\bar{w}, \bar{z})^*(y^*) - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*)+z_1^*)\big)\right], \quad (23)$$

for all $y^* \in \operatorname{int} \mathbb{R}^s_+$.

Proof Since $\tilde{\mathcal{F}}(\bar{w}) \subset \tilde{F}(w)$ for all $w \in W$ and the domination property holds for $\tilde{F}$ around $\bar{w}$, there exists a neighborhood V of $\bar{w}$ such that $\tilde{\mathcal{F}}(w)+K = \tilde{F}(w)+K$, $\forall w \in V$. So,

$$D^*_C(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*) = D^*_C(\tilde{F} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*),\ \forall y^* \in \mathbb{R}^s. \quad (24)$$

Since $\hat{\mathcal{F}}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$, $\tilde{\mathcal{F}}$ is Fréchet normally regular at $(\bar{w}, \bar{y})$ and assertion (ii) of Proposition 3.2, we get

$$D^*_C\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) \subset D^*_C(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*), \forall y^* \in \operatorname{int} \mathbb{R}^s_+. \quad (25)$$

Since the tangential regularity of $\tilde{F} + \mathbb{R}^s_+$ at $(\bar{w}, \bar{y})$, we can prove similarly to assertion (i) of Proposition 3.2 that

$$D^*_C(\tilde{F} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*) \subset D^*_C\tilde{F}(\bar{w}, \bar{y})(y^*), \forall y^* \in \mathbb{R}^s_+. \quad (26)$$

By the tangential regularity of $\tilde{F}$ at $(\bar{w}, \bar{y})$ and assertion (i) of Proposition 3.3,

$$D^*_C\tilde{F}(\bar{w}, \bar{y})(y^*) \subset \nabla_w f(\bar{w}, \bar{z})^*(y^*) - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\right], \quad (27)$$

for all $y^* \in \mathbb{R}^s$. Since (24)-(27), we obtain assertion (i) of theorem. We now prove assertion (ii). By the tangential regularity of $\tilde{\mathcal{F}} + \mathbb{R}^s_+$ at $(\bar{w}, \bar{y})$ and assertion (i) of Proposition 3.2, we get

$$D^*_C\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) \supset D^*_C(\tilde{\mathcal{F}} + \mathbb{R}^s_+)(\bar{w}, \bar{y})(y^*), \forall y^* \in \mathbb{R}^s. \quad (28)$$

Put a multifunction $\hat{F} : W \times \mathbb{R}^s \to \mathbb{R}^s$ defined by $\hat{F}(w, y) = \tilde{F}(w) \cap (y - \mathbb{R}^s_+)$. It is easy to see that $\hat{\mathcal{F}}(w, y) \subset \hat{F}(w, y)$ for all $(w, y) \in W \times \mathbb{R}^s$, dom $\hat{\mathcal{F}} =$ gph $(\tilde{\mathcal{F}}+\mathbb{R}^s_+)$ and dom $\hat{F} =$ gph $(\tilde{F} + \mathbb{R}^s_+)$. Combining this and the assumptions of theorem, we have that $\hat{F}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$. Combining this and Fréchet normal regularity of $\tilde{F}$ at $(\bar{w}, \bar{y})$, we can prove similarly to assertion (ii) of Proposition 3.2 that

$$D_C^*(\tilde{F} + \mathbb{R}_+^s)(\bar{w}, \bar{y})(y^*) \supset D_C^*\tilde{F}(\bar{w}, \bar{y})(y^*), \forall y^* \in \operatorname{int} \mathbb{R}_+^s. \tag{29}$$

By assertion (ii) of Proposition 3.3,

$$\begin{aligned} D_C^*\tilde{F}(\bar{w}, \bar{y})(y^*) \supset \nabla_w f(\bar{w}, \bar{z})^*(y^*) \\ - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\right], \end{aligned} \tag{30}$$

for all $y^* \in \operatorname{int} \mathbb{R}^s$. Combining (24) and (28)-(30), we obtain assertion (ii) of theorem. □

Let us give some illustrative examples for Theorem 3.1.

Example 3.1 Let $Z = \mathbb{R}^3$, $W = \mathbb{R}^2$, $\Omega = (0, +\infty) \times (0, +\infty) \times [\frac{\pi}{2}, \frac{3\pi}{2}]$, $K = \mathbb{R}_+^2$, $f(w, z) = \big(f^1(w, z), f^2(w, z)\big)$, where

$$\begin{aligned} f^1(w, z) &= \sqrt{2(z_1^2 + z_2^2)} - w_1 + w_2, \\ f^2(w, z) &= (z_1 - 1)^2 + (z_2 - 1)^2 - w_1 + w_2 \end{aligned}$$

and $G(w) = \big\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1+z_2 = 2w_1, \sin z_3 = 0\big\}$. Assume that $\bar{w} = (1, 0)$. Then one has $\bar{z} = (1, 1, \pi)$, $\bar{y} = f(\bar{w}, \bar{z}) = (1, -1)$ and

$$D_C^*\tilde{\mathcal{F}}(\bar{w}, \bar{y})(y_1^*, y_2^*) = \Big\{(y_1^* + y_2^*, -y_1^* + y_2^*)\Big\}, \ \forall y_1^*, y_2^* \in (0, +\infty).$$

Indeed, for $\bar{w} = (1, 0)$, we have the following problem

$$\operatorname{Min}_{\mathbb{R}_+^2}\Big\{\Big(\sqrt{2(z_1^2 + z_2^2)} - 1, (z_1 - 1)^2 + (z_2 - 1)^2 - 1\Big) : (z_1, z_2, z_3) \in G(\bar{w}) \cap \Omega\Big\},$$

where $G(\bar{w}) = \big\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_2 = 2, \sin z_3 = 0\big\}$. It is easy to check that $\bar{z} = (1, 1, \pi)$ is a solution of problem corresponding to $\bar{w}$ and therefore $\bar{y} = f(\bar{w}, \bar{z}) = \tilde{F}(\bar{w}) = \tilde{\mathcal{F}}(\bar{w}) = (1, -1)$, and $N(\bar{z}; \Omega) = 0_{\mathbb{R}^3}$. We have

$$\begin{aligned} G_\Omega(w) &= \{z \in \Omega : z_1 + z_2 = 2w_1, \ \sin z_3 = 0\} \\ &= \{z_1, z_2 > 0, z_3 = \pi : z_1 + z_2 = 2w_1\}, \\ \operatorname{gph} G_\Omega &= \{(w, z) \in W \times \Omega : z_1 + z_2 = 2w_1, \ \sin z_3 = 0\} \\ &= \{(w, z) \in \mathbb{R}^5 : w_1, z_1, z_2 > 0, z_3 = \pi, z_1 + z_2 = 2w_1\} \end{aligned}$$

and

$$\tilde{F}(w) = \Big\{y = (y_1, y_2) = \big(f^1(w, z), f^2(w, z)\big) : z \in G_\Omega(w)\Big\}$$

$$
\begin{aligned}
&= \Big\{ y = (y_1, y_2) : y_1 = 2\sqrt{(z_1 - w_1)^2 + w_1^2} - w_1 + w_2, \\
&\qquad y_2 = 2(z_1 - w_1)^2 + 2w_1^2 + w_2 - w_1 + 2 : z_1 > 0 \Big\} \\
&= \Big\{ y = (y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 2|w_1| - w_1 + w_2,\ y_2 \geq 2w_1^2 + w_2 - w_1 + 2 \Big\} \\
&= \tilde{\mathcal{F}}(w) + \mathbb{R}^2_+.
\end{aligned}
$$

So, the domination property holds for $\tilde{F}$ around $\bar{w}$ and $\hat{\mathcal{F}}(w, y) = \tilde{\mathcal{F}}(w) \cap (y - \mathbb{R}^2_+)$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$. We also get

$$
\begin{aligned}
\operatorname{gph} \tilde{F} = \Big\{ (w, y) = (w_1, w_2, y_1, y_2) \in \mathbb{R}^4 : y_1 = 2\sqrt{(z_1 - w_1)^2 + w_1^2} - w_1 + w_2, \\
y_2 = 2(z_1 - w_1)^2 + 2w_1^2 + w_2 - w_1 + 2, \\
w_1, z_1 > 0 \Big\}.
\end{aligned}
$$

So, we can show that $\tilde{F}, \tilde{\mathcal{F}}$ and $\tilde{F} + \mathbb{R}^2_+, \tilde{\mathcal{F}} + \mathbb{R}^2_+$ are normally regular at $(\bar{w}, \bar{y})$. We also prove that the mapping $\tilde{G}_\Omega$, which is defined by

$$
\begin{aligned}
\tilde{G}_\Omega(w, y) &= \{ z \in G_\Omega(w) : y = f(w, z) \} \\
&= \Big\{ z_1, z_2 > 0, z_3 = \pi : z_1 + z_2 = 2w_1, y_1 = 2\sqrt{(z_1 - w_1)^2 + w_1^2} - w_1 + w_2, \\
&\qquad y_2 = 2(z_1 - w_1)^2 + 2w_1^2 + w_2 \\
&\qquad - w_1 + 2 \Big\},
\end{aligned}
$$

admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{z})$. It is easy to see that

$$
\nabla_w f(\bar{w}, \bar{z}) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \nabla_z f(\bar{w}, \bar{z}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

and

$$
\nabla_w g(\bar{w}, \bar{z})(w_1, w_2) = (-2w_1, 0),\ \nabla_z g(\bar{w}, \bar{z})(z_1, z_2, z_3) = (z_1 + z_2, -z_3).
$$

Thus, assumptions of Theorem 3.1 are satisfied. By this theorem,

$$
\begin{aligned}
D_C^* \tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) &= \nabla_w f(\bar{w}, \bar{z})^*(y^*) \\
&\quad - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \Big[\nabla_w g(\bar{w}, \bar{z})^* \big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1} (\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*) \big) \Big],
\end{aligned}
\tag{31}
$$

for all $y^* = (y_1^*, y_2^*) \in (0, +\infty) \times (0, +\infty)$. Note that for any $y^* = (y_1^*, y_2^*) \in (0, +\infty) \times (0, +\infty)$, we have

$$\nabla_w f(\bar{w}, \bar{z})^*(y^*) = (-y_1^* + y_2^*, -y_1^* + y_2^*),\ \nabla_z f(\bar{w}, \bar{z})^*(y^*) = (y_1^*, y_1^*, 0),\ N(\bar{z}; \Omega) = \{0\}.$$

So, $(\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*) = (y_1^*, 0)$,

$$-\bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^* \big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(y^*) + z_1^*)\big)\right] = (2y_1^*, 0).$$

Combining this and (31), we obtain

$$D_C^* \tilde{\mathcal{F}}(\bar{w}, \bar{y})(y^*) = \left\{(y_1^* + y_2^*, -y_1^* + y_2^*)\right\},\ \forall y^* = (y_1^*, y_2^*) \in (0, +\infty) \times (0, +\infty).$$

The following example shows that assumptions in Theorem 3.1 are essential. Particularly, inclusion (23) may fail to hold if the assumption on the existence of the local upper Lipschitzian selection of $\tilde{G}_\Omega$ at the point under consideration is omitted.

Example 3.2 Let $W = Z = \mathbb{R}$, $s = 2$, $\Omega = [-1, +\infty)$, $f(w, z) = (z^2, z^2 + w)$ and $G(w) = \left\{z \in \mathbb{R} : z^2 - w = 0\right\}$. Assume that $\bar{w} = 0$. Then one has $\bar{z} = 0$, $\bar{y} = f(\bar{w}, \bar{z}) = (0, 0)$ and $D_C^* \tilde{\mathcal{F}}(\bar{w}, \bar{y})(1, 1) = (-\infty, 0]$. While

$$\nabla_w f(\bar{w}, \bar{z})^*(1, 1) - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^* \big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(1, 1) + z_1^*)\big)\right] = \mathbb{R}.$$

Indeed, for $\bar{w} = 0$, we have $\tilde{\mathcal{F}}(\bar{w}) = \mathrm{Min}_{\mathbb{R}^2_+}\{(z^2, z^2) : z \in G_\Omega(\bar{w})\}$, where $G_\Omega(\bar{w}) = \left\{z \in \mathbb{R} : z^2 = 0, z \geq -1\right\}$. It is easy to check that $\bar{z} = 0$ is the unique solution of the problem corresponding to $\bar{w}$ and therefore $\bar{y} = f(\bar{w}, \bar{z}) = \tilde{F}(\bar{w}) = \tilde{\mathcal{F}}(\bar{w}) = (0, 0)$ and $N(\bar{z}; \Omega) = 0$. We have

$$G(w) = \begin{cases} \{\sqrt{w}, -\sqrt{w}\} & \text{if } w \geq 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

$$G_\Omega(w) = \begin{cases} \{\sqrt{w}\} & \text{if } w > 1 \\ \{\sqrt{w}, -\sqrt{w}\} & \text{if } 0 \leq w \leq 1 \\ \emptyset & \text{if } w < 0, \end{cases}$$

$$\mathrm{gph}\, G_\Omega(w) = \begin{cases} \{(w, \sqrt{w})\} & \text{if } w > 1 \\ \{(w, \sqrt{w}), (w, -\sqrt{w})\} & \text{if } 0 \leq w \leq 1 \\ \emptyset & \text{if } w < 0 \end{cases}$$

and

$$\tilde{F}(w) = \{y = f(w, z) : z \in G_\Omega(w)\} = \{(w, 2w) : w \geq 0\}.$$

So, the domination property holds for $\tilde{F}$ around $\bar{w}$ and $\hat{\mathcal{F}}(w, y) = \tilde{\mathcal{F}}(w) \cap (y - \mathbb{R}^2_+)$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$. We also get

$$\text{gph } \tilde{F} = \{(w, y) = (w, y_1, y_2) \in \mathbb{R}^3 : w \geq 0,\ y_1 = w,\ y_2 = 2w\}.$$

So, we can show that $\tilde{F}, \tilde{\mathcal{F}}$ and $\tilde{F} + \mathbb{R}_+, \tilde{\mathcal{F}} + \mathbb{R}_+$ are normally regular at $(\bar{w}, \bar{y})$. We also prove that the mapping $\tilde{G}_\Omega$, which is defined by

$$\tilde{G}_\Omega(w, y) = \begin{cases} \{\sqrt{w}\} & \text{if } w > 1, y = (w, 2w) \\ \{\sqrt{w}, -\sqrt{w}\} & \text{if } 0 \leq w \leq 1, y = (w, 2w) \\ \emptyset & \text{if } w < 0 \end{cases}$$

does not admit a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{z})$. It is easy to see that $\nabla_w f(\bar{w}, \bar{z}) = (0, 1),\ \nabla_z f(\bar{w}, \bar{z}) = (0, 0)$ and $\nabla_w g(\bar{w}, \bar{z}) = -1,\ \nabla_z g(\bar{w}, \bar{z}) = 0$. Thus, the remaining assumptions of Theorem 3.1 are satisfied. We are able to calculate directly, $\nabla_z f(\bar{w}, \bar{z})^*(1, 1) + z_1^* = 0, \forall z_1^* \in \hat{N}(\bar{z}, \Omega)$. So, $\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(1, 1) + z_1^*)\big) = \mathbb{R}$. Hence,

$$\nabla_w f(\bar{w}, \bar{z})^*(1, 1) - \bigcup_{z_1^* \in \hat{N}(\bar{z}, \Omega)} \left[\nabla_w g(\bar{w}, \bar{z})^*\big((\nabla_z g(\bar{w}, \bar{z})^*)^{-1}(\nabla_z f(\bar{w}, \bar{z})^*(1, 1) + z_1^*)\big)\right] = \mathbb{R}.$$

While, $D^*_C \tilde{\mathcal{F}}(\bar{w}, \bar{y})(1, 1) = (-\infty, 0]$.

4 Sensitivity Analysis in Multi-objective Optimal Control Problems

Based on Theorem 3.1, we can obtain formulae for upper and lower-evaluation on the Clarke coderivatives of the extremum multifunction $\mathcal{F}$ in the multi-objective parametric optimal control problem (1)–(4).

To deal with our problem, we impose the following assumptions:
$(A1)$ The functions $L^i : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \bar{\mathbb{R}}$ and $g^i : \mathbb{R}^n \to \bar{\mathbb{R}}$ $(i = 1, 2, \ldots, s)$ have the properties that $L^i(\cdot, x, u, v)$ is measurable for all $(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$, $L^i(t, \cdot, \cdot, \cdot)$ and $g^i(\cdot)$ are continuously differentiable for almost every $t \in [0, 1]$, and there exist constants $c_1 > 0, c_2 > 0, r \geq 0$, a nonnegative function $\omega_1 \in L^p([0, 1], \mathbb{R})$, constants $0 \leq p_1 \leq p, 0 \leq p_2 \leq p - 1$ such that

$$\begin{aligned} &|L^i(t, x, u, v)| \leq c_1\big(\omega_1(t) + |x|^{p_1} + |u|^{p_1} + |v|^{p_1}\big), \\ &\max\left\{|L^i_x(t, x, u, v)|, |L^i_u(t, x, u, v)|, |L^i_v(t, x, u, v)|\right\} \leq c_2\big(|x|^{p_2} + |u|^{p_2} \\ &+ |v|^{p_2}\big) + r \end{aligned}$$

for all $(t, x, u, v) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k$.
$(A2)$ The matrix-valued functions $B : [0, 1] \to M_{n,m}(\mathbb{R})$ and $T : [0, 1] \to M_{n,k}(\mathbb{R})$ are measurable and essentially bounded.

$(A3)$ The function $\varphi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ has properties that $\varphi(t, \cdot)$ is of class C^1 for almost every $t \in [0,1]$, $\varphi(\cdot, 0) \in L^p([0,1], \mathbb{R}^n)$ and for each $M > 0$, there exists a positive number $l_{\varphi M}$ such that

$$| \varphi_x(t,x) | \leq l_{\varphi M}, \quad | \varphi_x(t,x_1) - \varphi_x(t,x_2) | \leq l_{\varphi M}|x_1 - x_2|,$$

for a.e. t $\in [0,1]$, for all $x, x_1, x_2 \in \mathbb{R}^n$ satisfying $|x|, |x_1|, |x_2| \leq M$.

In the notation of Subsections 2.1, put $V = L^p([0,1], \mathbb{R}^n)$ and

$$h : X \times U \times W \to V \times \mathbb{R}^n$$

defined by

$$h(x,u,w) = \big(h_1(x,u,w), h_2(x,u,w)\big) := \big(\dot{x} - \varphi(\cdot, x) - Bu - T\theta, x(0) - \alpha\big).$$

Under the hypotheses $(A2)$–$(A3)$, (5) can be written in the form

$$H(w) = \Big\{(x,u) \in X \times U : h(x,u,w) = 0\Big\}.$$

Consider the multifunctions $\hat{\mathcal{F}} : W \times \mathbb{R}^s \rightrightarrows \mathbb{R}^s$ and $\tilde{H}_K : W \times \mathbb{R}^s \rightrightarrows Z$ as follows $\hat{\mathcal{F}}(w,y) = \mathcal{F}(w) \cap (y - \mathbb{R}^s_+)$ and $\tilde{H}_K(w,y) = \{z \in H_K(w) : y = J(z,w) = J(x,u,w)\}$. We are now ready to state our main result.

Theorem 4.1 *Let $\mathcal{U}$ be locally closed around $\bar{u}$, epi-Lipschitzian at $\bar{u}$ and $\bar{w} = (\bar{\alpha}, \bar{\theta}) \in W$, $\bar{z} = (\bar{x}, \bar{u}) \in H_K(\bar{w}) = H(\bar{w}) \cap K$ be such that $\bar{y} = f(\bar{w}, \bar{z}) \in \tilde{\mathcal{F}}(\bar{w})$. Suppose that assumptions $(A1)$–$(A3)$ and the following regularity conditions are satisfied*

$$\Big\{(\alpha, \theta, x, u) \in \mathbb{R}^n \times \Theta \times X \times U : \dot{x} - \varphi_x(\cdot, \bar{x})x - Bu - T\theta = 0, x(0) = \alpha\Big\}$$
$$\cap \Big[\mathbb{R}^n \times \Theta \times X \times \operatorname{int} T(\bar{z}, \mathcal{U})\Big] \neq \emptyset. \qquad (32)$$

Assume further that the domination property holds for F around $\bar{w}$ and $\hat{\mathcal{F}}$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$.

(i) Suppose that F, $F + \mathbb{R}^s_+$ are tangentially regular at $(\bar{w}, \bar{y})$ and $\mathcal{F}$ is Fréchet normally regular at $(\bar{w}, \bar{y})$. Then for a vector $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0,1], \mathbb{R}^k)$, $(\alpha^*, \theta^*) \in D^*_C\mathcal{F}(\bar{w}, \bar{y})(y^*)$ with $y^* = (y_1^*, y_2^*, \ldots, y_s^*) \in \operatorname{int} \mathbb{R}^s_+$, it is necessary that there exist functions $y \in W^{1,q}([0,1], \mathbb{R}^n)$ and $u^* \in L^q([0,1], \mathbb{R}^m)$ with $u^* \in N(\bar{u}, \mathcal{U})$ such that the following conditions are satisfied:

$$\alpha^* = \sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big)y_i^* + \sum_{i=1}^{s} \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)y_i^* dt$$
$$- \int_0^1 \varphi_x\big(t, \bar{x}(t)\big)y(t)dt,$$

$$y(1) = -\sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big) y_i^*,$$

and

$$\Big(\dot{y}(t) + \varphi_x\big(t, \bar{x}(t)\big)y(t), B^T(t)y(t) - u^*(t), \theta^*(t) + T^T(t)y(t)\Big)$$
$$= \sum_{i=1}^{s} \nabla L^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big) y_i^* \text{ a.e. } t \in [0, 1]. \tag{33}$$

The above conditions are also sufficient for $(\alpha^*, \theta^*) \in D_C^*\mathcal{F}(\bar{w}, \bar{y})(y^*)$ *with* $y^* = (y_1^*, y_2^*, \ldots, y_s^*) \in \operatorname{int} \mathbb{R}_+^s$ *if* $\mathcal{F} + \mathbb{R}_+^s$ *is tangentially regular at* $(\bar{w}, \bar{y})$*,* F *is Fréchet normally regular at this point,* H_K *is Fréchet normally regular at* $(\bar{w}, \bar{z})$ *and* $\tilde{H}_K$ *admits a local upper Lipschitzian selection at* $(\bar{w}, \bar{y}, \bar{z})$*. Here,* B^T *stands for the transpose of* B*,* $\nabla L^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)$ *stands for the gradient of* $L^i(t, \cdot, \cdot, \cdot)$ *at* $\big(\bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)$ *and* q *is the conjugate number of* p*, that is,* $1 < q < +\infty$ *and* $1/p + 1/q = 1$*.*

When $\mathcal{U} = U$ or $\bar{u} \in \operatorname{int} \mathcal{U}$, we obtain the following corollary.

Corollary 4.1 *Let* $\bar{w} = (\bar{\alpha}, \bar{\theta}) \in W$*,* $\bar{z} = (\bar{x}, \bar{u}) \in H(\bar{w})$ *be such that* $\bar{y} = f(\bar{w}, \bar{z}) \in \tilde{\mathcal{F}}(\bar{w})$ *and assumptions* $(A1)$*–*$(A3)$ *be satisfied. Assume further that the domination property holds for* F *around* $\bar{w}$ *and* $\hat{\mathcal{F}}$ *admits a local upper Lipschitzian selection at* $(\bar{w}, \bar{y}, \bar{y})$*.*

(i) Suppose that F*,* $F + \mathbb{R}_+^s$ *are tangentially regular at* $(\bar{w}, \bar{y})$ *and* $\mathcal{F}$ *is Fréchet normally regular at* $(\bar{w}, \bar{y})$*. Then for a vector* $(\alpha^*, \theta^*) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^k)$*,* $(\alpha^*, \theta^*) \in D_C^*\mathcal{F}(\bar{w}, \bar{y})(y^*)$ *with* $y^* = (y_1^*, y_2^*, \ldots, y_s^*) \in \operatorname{int} \mathbb{R}_+^s$*, it is necessary that there exists a function* $y \in W^{1,q}([0, 1], \mathbb{R}^n)$ *such that the following conditions are satisfied:*

$$\alpha^* = \sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big) y_i^* + \sum_{i=1}^{s} \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big) y_i^* dt$$
$$- \int_0^1 \varphi_x\big(t, \bar{x}(t)\big) y(t) dt,$$
$$y(1) = -\sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big) y_i^*,$$

and

$$\Big(\dot{y}(t) + \varphi_x\big(t, \bar{x}(t)\big)y(t), B^T(t)y(t), \theta^*(t) + T^T(t)y(t)\Big)$$
$$= \sum_{i=1}^{s} \nabla L^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big) y_i^* \text{ a.e. } t \in [0, 1]. \tag{34}$$

The above conditions are also sufficient for $(\alpha^*, \theta^*) \in D_C^*\mathcal{F}(\bar{w}, \bar{y})(y^*)$ *with* $y^* = (y_1^*, y_2^*, \ldots, y_s^*) \in \text{int}\ \mathbb{R}_+^s$ *if* $\mathcal{F} + \mathbb{R}_+^s$ *is tangentially regular at* $(\bar{w}, \bar{y})$*, F is Fréchet normally regular at this point, H is Fréchet normally regular at* $(\bar{w}, \bar{z})$ *and* $\tilde{H}$ *admits a local upper Lipschitzian selection at* $(\bar{w}, \bar{y}, \bar{z})$*.*

Recall that for $1 < p < \infty$, we have $L^p([0, 1], \mathbb{R}^n)^* = L^q([0, 1], \mathbb{R}^n)$, where

$$1 < q < +\infty, \quad 1/p + 1/q = 1.$$

Besides, $L^p([0, 1], \mathbb{R}^n)$ is pared with $L^q([0, 1], \mathbb{R}^n)$ by the formula

$$\langle x^*, x\rangle = \int_0^1 \langle x^*(t), x(t)\rangle dt,$$

for all $x^* \in L^q([0, 1], \mathbb{R}^n)$ and $x \in L^p([0, 1], \mathbb{R}^n)$.

Also, we have $W^{1,p}([0, 1], \mathbb{R}^n)^* = \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ and $W^{1,p}([0, 1], \mathbb{R}^n)$ is pared with $\mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ by the formula

$$\langle (a, u), x\rangle = \langle a, x(0)\rangle + \int_0^1 \langle u(t), \dot{x}(t)\rangle dt,$$

for all $(a, u) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ and $x \in W^{1,p}([0, 1], \mathbb{R}^n)$ (see [15, p. 21]).

In the case of $p = 2$, $W^{1,2}([0, 1], \mathbb{R}^n)$ becomes a Hilbert space with the inner product given by

$$\langle x, y\rangle = \langle x(0), y(0)\rangle + \int_0^1 \langle \dot{x}(t), \dot{y}(t)\rangle dt,$$

for all $x, y \in W^{1,2}([0, 1], \mathbb{R}^n)$.

Given $x \in X$, we put $M = \|x\|_0 = \max_{t\in[0,1]} |x(t)|$. By assumption $(A3)$, there exists a constant $l_{\varphi M} > 0$ such that $|\varphi_x(t, x)| \le l_{\varphi M}$ for a.e.$t \in [0, 1]$, for all $x \in \mathbb{R}^n$ satisfying $|x| \le M$. By the Taylor expansion, we get

$$\begin{aligned} |\varphi(t, x(t))| &\le |\varphi(t, x(t)) - \varphi(t, 0)| + |\varphi(t, 0)| \\ &= |\varphi_x\big(t, \theta(t)x(t)\big)x(t)| + |\varphi(t, 0)| \\ &\le l_{\varphi M} M + |\varphi(t, 0)|. \end{aligned}$$

This implies that $\varphi(\cdot, x) \in L^p([0, 1], \mathbb{R}^n)$.

Using the similar technique as in the proof of [36, Lemma 7], we obtain the following result.

Lemma 4.1 *Suppose that assumptions* $(A2)$–$(A3)$ *are valid. Then, function h is differentiable around* $(\bar{z}, \bar{w}) = (\bar{x}, \bar{u}, \bar{\alpha}, \bar{\theta})$*,* ∇h *is continuous at* $(\bar{z}, \bar{w})$ *and*

$$\nabla_z h(\bar{z}, \bar{w})^*(u^*, a) = \Big(a - \int_0^1 u^*(t)\varphi_x\big(t, \bar{x}(t)\big)dt, u^*$$

$$
- \int_{(\cdot)}^{1} u^*(\tau)\varphi_x\big(\tau, \bar{x}(\tau)\big)d\tau, -Bu^*\Big),
$$
$$
\nabla_w h(\bar{z}, \bar{w})^*(u^*, a) = \Big(-a, -Tu^*\Big),
$$

for any $u^* \in L^q([0,1], \mathbb{R}^n)$ *and any* $a \in \mathbb{R}^n$.

Recall that our problem can be written in the form

$$
\mathrm{Min}_{\mathbb{R}^s_+} \quad J(z, w), \quad \text{subject to } \; z \in H(w) \cap K.
$$

In the sequel, we shall need the following lemmas.

Lemma 4.2 ([34, Lemma 3.1]) *Suppose that assumption* $(A1)$ *is valid. Then, the function* J *is strictly differentiable at* $(\bar{z}, \bar{w})$ *and* $\nabla J(\bar{z}, \bar{w})$ *is given by*

$$
\begin{aligned}
&\nabla_w J(\bar{z}, \bar{w}) = \Big(\nabla_w J^1(\bar{z}, \bar{w}), \nabla_w J^2(\bar{z}, \bar{w}), \ldots, \nabla_w J^s(\bar{z}, \bar{w})\Big)^T,\\
&\nabla_w J^i(\bar{z}, \bar{w}) = \Big(0, L^i_\theta\big(\cdot, \bar{x}, \bar{u}, \bar{\theta}\big)\Big), \; i = 1, 2, \ldots, s,\\
&\nabla_z J(\bar{z}, \bar{w}) = \Big(\nabla_z J^1(\bar{z}, \bar{w}), \nabla_z J^2(\bar{z}, \bar{w}), \ldots, \nabla_z J^s(\bar{z}, \bar{w})\Big),\\
&\nabla_z J^i(\bar{z}, \bar{w}) = \big(J^i_x(\bar{x}, \bar{u}, \bar{\theta}), J^i_u(\bar{x}, \bar{u}, \bar{\theta})\big), \; i = 1, 2, \ldots, s,
\end{aligned}
$$

with

$$
J^i_u(\bar{x}, \bar{u}, \bar{\theta}) = L^i_u(\cdot, \bar{x}, \bar{u}, \bar{\theta})
$$

and

$$
\begin{aligned}
J^i_x(\bar{x}, \bar{u}, \bar{\theta}) = \Big(&(g^i)'\big(x(1)\big) + \int_0^1 L^i_x\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)dt,\\
&(g^i)'\big(x(1)\big) + \int_{(\cdot)}^1 L^i_x\big(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau)\big)d\tau\Big).
\end{aligned}
$$

We have

$$
\nabla_z h(\bar{z}, \bar{w})z = \big(\dot{x} - \varphi_x(\bar{x})x - Bu, x(0)\big).
$$

Using the similar technique as in the proof of [15, Corollary p. 52], we obtain the following result.

Lemma 4.3 *Suppose that assumptions* $(A2)$–$(A3)$ *are valid. Then,* $\nabla_z h(\bar{z}, \bar{w})$ *is surjective.*

We now return to the proof of Theorem 4.1, our main result.

By Lemma 4.1, $h(x, u, \alpha, \theta)$ is differentiable around $(\bar{x}, \bar{u}, \bar{\alpha}, \bar{\theta})$, ∇h is continuous at $(\bar{x}, \bar{u}, \bar{\alpha}, \bar{\theta})$ and

$$\nabla h\big((\bar{\alpha}, \bar{\theta}), (\bar{x}, \bar{u})\big)\big((\alpha, \theta), (x, u)\big) = \big(\dot{x} - \varphi_x(\cdot, \bar{x})x - Bu - T\theta, x(0) - \alpha\big).$$

By (32), the condition (22) is satisfied. Since Lemma 4.2, the function J is Fréchet differentiable at $(\bar{x}, \bar{u}, \bar{\alpha}, \bar{\theta})$. Thus, the assumptions of Theorem 3.1 are fulfilled. Theorem 3.1 follows that if $y^* = (y_1^*, y_2^*, \ldots, y_s^*) \in \operatorname{int} \mathbb{R}_+^s$ and $w^* = (\alpha^*, \theta^*) \in D_C^*\mathcal{F}(\bar{w}, \bar{y})(y^*)$ then there exist a function $z^* \in N(\bar{z}; K)$ and $v^* = (a, v) \in \mathbb{R}^n \times L^q([0, 1], \mathbb{R}^n)$ such that

$$\begin{aligned} &w^* = \nabla_w J(\bar{z}, \bar{w})^T(y^*) - \nabla_w h(\bar{z}, \bar{w})^* v^* \text{ and} \\ &\nabla_z J(\bar{z}, \bar{w})^T(y^*) + z^* = \nabla_z h(\bar{z}, \bar{w})^* v^*. \end{aligned} \tag{35}$$

It is easy to see that $z^* = (0, u^*)$ for some $u^* \in N(\bar{u}; \mathcal{U})$. Since Lemma 4.2, we have

$$\nabla_w J(\bar{z}, \bar{w})^T(y^*) = \sum_{i=1}^{s} \nabla_w J^i(\bar{z}, \bar{w}) y_i^* \ \text{ and } \ \nabla_z J(\bar{z}, \bar{w})^T(y^*) = \sum_{i=1}^{s} \nabla_z J^i(\bar{z}, \bar{w}) y_i^*.$$

Combining this and the equation (35), we have

$$\begin{aligned} &\Big(\alpha^*, \theta^* - \sum_{i=1}^{s} J_\theta^i(\bar{z}, \bar{w}) y_i^*\Big) = -\nabla_w h(\bar{z}, \bar{w})^*(a, v) \text{ and} \\ &\sum_{i=1}^{s} \nabla_z J^i(\bar{z}, \bar{w}) y_i^* + z^* = \nabla_z h(\bar{z}, \bar{w})^*(a, v). \end{aligned} \tag{36}$$

Combining this and Lemmas 4.1, 4.2, we get

$$(36) \Leftrightarrow \begin{cases} \alpha^* = a; \quad \theta^* - \sum_{i=1}^{s} J_\theta^i(\bar{z}, \bar{w}) y_i^* = T^T(\cdot)v(\cdot) \\ \big(\sum_{i=1}^{s} J_x^i(\bar{z}, \bar{w}) y_i^*, \sum_{i=1}^{s} J_u^i(\bar{z}, \bar{w}) y_i^* + u^*\big) \\ = \big(\nabla_x h(\bar{z}, \bar{w})^*(a, v), \nabla_u h(\bar{z}, \bar{w})^*(a, v)\big). \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha^* = a \\ \theta^* = \sum_{i=1}^{s} L_\theta^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) y_i^* + T^T(\cdot)v(\cdot) \\ \sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big) y_i^* + \sum_{i=1}^{s} \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big) y_i^* dt \\ = a - \int_0^1 \varphi_x(t, \bar{x}(t)) v(t) dt \\ \sum_{i=1}^{s} (g^i)'\big(\bar{x}(1)\big) y_i^* + \sum_{i=1}^{s} \int_{(\cdot)}^1 L_x^i\big(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau)\big) y_i^* d\tau \\ = v(\cdot) - \int_{(\cdot)}^1 \varphi_x(\tau, \bar{x}(\tau)) v(\tau) d\tau \\ \sum_{i=1}^{s} L_u^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big) y_i^* + u^* = -B^T(\cdot)v(\cdot). \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha^* = a \\ \theta^* = \sum_{i=1}^s L_\theta^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big)y_i^* + T^T(\cdot)v(\cdot) \\ \sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* + \sum_{i=1}^s \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)y_i^* dt \\ = a - \int_0^1 \varphi_x(t, \bar{x}(t))v(t)dt \\ \sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* - \sum_{i=1}^s \int_1^{(\cdot)} L_x^i\big(\tau, \bar{x}(\tau), \bar{u}(\tau), \bar{\theta}(\tau)\big)y_i^* d\tau \\ = v(\cdot) + \int_1^{(\cdot)} \varphi_x(\tau, \bar{x}(\tau))v(\tau)d\tau \\ \sum_{i=1}^s L_u^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big)y_i^* + u^* = -B^T(\cdot)v(\cdot). \end{cases}$$

$$\Leftrightarrow \begin{cases} v \in W^{1,q}([0,1], R^n) \\ \theta^* - T^T(\cdot)v(\cdot) = \sum_{i=1}^s L_\theta^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big)y_i^* \\ \alpha^* = \sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* + \sum_{i=1}^s \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)y_i^* dt \\ + \int_0^1 \varphi_x(t, \bar{x}(t))v(t)dt \\ v(1) = \sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* \\ -\dot{v}(\cdot) - \varphi_x(\cdot, \bar{x}(\cdot)v(\cdot) = \sum_{i=1}^s L_x^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big)y_i^* \\ -B^T(\cdot)v(\cdot) = \sum_{i=1}^s L_u^i\big(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{\theta}(\cdot)\big)y_i^* + u^*. \end{cases} \tag{37}$$

Putting $y = -v$, we obtain

$$(37) \Leftrightarrow \begin{cases} \alpha^* = \sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* + \sum_{i=1}^s \int_0^1 L_x^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)y_i^* dt \\ - \int_0^1 \varphi_x(t, \bar{x}(t))y(t)dt \\ y(1) = -\sum_{i=1}^s (g^i)'\big(\bar{x}(1)\big)y_i^* \\ \big(\dot{y}(t) + \varphi_x(t, \bar{x}(t)y(t), B^T(t)y(t) - u^*(t), \theta^*(t) + T^T(t)y(t)\big) \\ = \sum_{i=1}^s \nabla L^i\big(t, \bar{x}(t), \bar{u}(t), \bar{\theta}(t)\big)y_i^*, \end{cases}$$

for a.e. $t \in [0, 1]$. This is the first assertion of theorem. Using the second conclusion of Theorem 3.1, we also obtain the second assertion of theorem. The proof of Theorem 4.1 is complete. □

To illustrate Theorem 4.1, we provide the following example.

Example 4.1 Let $X = W^{1,2}([0,1], \mathbb{R}^2)$, $U = L^2([0,1], \mathbb{R}^2)$, $\Theta = L^2([0,1], \mathbb{R}^2)$, $W = \mathbb{R}^2 \times \Theta$. Consider the problem $\mathrm{Min}_{\mathbb{R}^2_+} J(x, u, w)$

$$\text{subject to } \begin{cases} \dot{x}_1 = t + 2x_1 + u_1 + \theta_1, & \dot{x}_2 = \sin x_2, \\ x_1(0) = \alpha_1, & x_2(0) = \alpha_2, \end{cases}$$

where $J(x, u, w) = \big(J^1(x, u, w), J^2(x, u, w)\big)$,

$$J^1(x, u, w) = \int_0^1 \Big(u_1^2 + \frac{1}{1 + u_1^2} + u_2^2 + \theta_1^2\Big)dt$$

and

$$J^2(x, u, w) = \int_0^1 \big(u_1^2 + u_2^2 + \theta_2^2\big)dt.$$

Then, for $\bar{w} = (\bar{\alpha}, \bar{\theta}), \bar{\alpha} = (1, 0), \bar{\theta} = (0, 0),\ \bar{x} = \Big(\frac{5}{4}e^{2t} - \frac{t}{2} - \frac{1}{4}, 0\Big),\ \bar{u} = (0, 0)$ and $\bar{y} = J(\bar{w}, \bar{x}, \bar{u}) = (1, 0)$, we have

$$D_C^*\mathcal{F}(\bar{w}, \bar{y})(y^*) \subset \{(0_{\mathbb{R}^2}, 0_{L^2([0,1],\mathbb{R}^2)})\}, \quad y^* = (y_1^*, y_2^*) \in \operatorname{int} \mathbb{R}^2_+.$$

In [35, Example 3.1], it was shown that assumption $(A1)$ is satisfied. It is easy to show that assumptions $(A2)$–$(A3)$ are also satisfied. We have

$$\begin{aligned} H(w) = \Big\{&(x, u) = \big((x_1, x_2), (u_1, u_2)\big) \in X \times U :\ \dot{x}_1 - 2x_1 - t - u_1 \\ &- \theta_1 = 0, x_1(0) = \alpha_1; x_2 = 2\arctan\big(\tan(\frac{\alpha_2}{2})e^t\big)\Big\}, \\ \text{gph } H = \Big\{&(w, z) \in W \times Z :\ w = (\alpha, \theta), z = (x, u), \\ &\alpha = (\alpha_1, \alpha_2), \theta = (\theta_1, \theta_2), x = (x_1, x_2), u = (u_1, u_2), \\ &\dot{x}_1 - 2x_1 - t - u_1 - \theta_1 = 0, x_1(0) = \alpha_1; \\ &x_2 = 2\arctan\big(\tan(\frac{\alpha_2}{2})e^t\big)\Big\}. \end{aligned}$$

It is also easy to check that

$$\begin{aligned} F(w) &= \Big\{y = J(w, z) : z \in H(w)\Big\} \\ &= \Big\{y = \big(J^1(x, u, w), J^2(x, u, w)\big) : z = (x, u) \in H(w)\Big\} \\ &\subset \operatorname{Min}_{\mathbb{R}^2_+} F(w) + \mathbb{R}^2_+ \\ &= [1 + \|\theta_1\|^2, +\infty) \times [\|\theta_2\|^2, +\infty) \end{aligned}$$

and

$$\mathcal{F}(w) = \Big(1 + \int_0^1 \theta_1^2(t)dt, \int_0^1 \theta_2^2(t)dt\Big)$$

for all $w = (\alpha, \theta) \in W, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \theta = (\theta_1, \theta_2) \in L^2([0, 1], \mathbb{R}^2)$. So, the domination property holds for F around $\bar{w}$ and $\hat{\mathcal{F}}(w, y) = \mathcal{F}(w) \cap (y - \mathbb{R}^2_+)$ admits a local upper Lipschitzian selection at $(\bar{w}, \bar{y}, \bar{y})$. We also get

$$\text{gph } F = \Big\{(w, y) \in W \times \mathbb{R}^2 : y = \big(J^1(x, u, w), J^2(x, u, w)\big), z = (x, u) \in H(w)\Big\}.$$

So, we can show that F, $F + \mathbb{R}^2_+$ are tangentially regular and $\mathcal{F}$ are normally regular at $(\bar{w}, \bar{y})$. Thus, all assumptions of Corollary 4.1 are satisfied. Take any $y^* = (y_1^*, y_2^*) \in$

int $\mathbb{R}^2_+$ and $w^* = (\alpha^*, \theta^*) \in D^*_C\mathcal{F}(\bar{w}, \bar{y})(y^*)$. By Corollary 4.1 there exists $y = (y_1, y_2) \in W^{1,2}([0,1], \mathbb{R}^2)$ such that

$$\begin{cases} \alpha^* = \big(\alpha_1^*, \alpha_2^*\big) = \big(\int_0^1 2y_1(t)dt, \int_0^1 y_2(t)dt\big) \\ y_1(1) = 0, \quad y_2(1) = 0 \\ \dot{y}(t) + \varphi_x(t, \bar{x}(t))y(t) = 0 \\ B^T y(t) = 0 \\ \theta^*(t) = -T^T(t)y(t). \end{cases}$$

This is equivalent to $\alpha^* = (0, 0)$ and $\theta^* = (0, 0)$.

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Declarations

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